SOLID GEOMETRY

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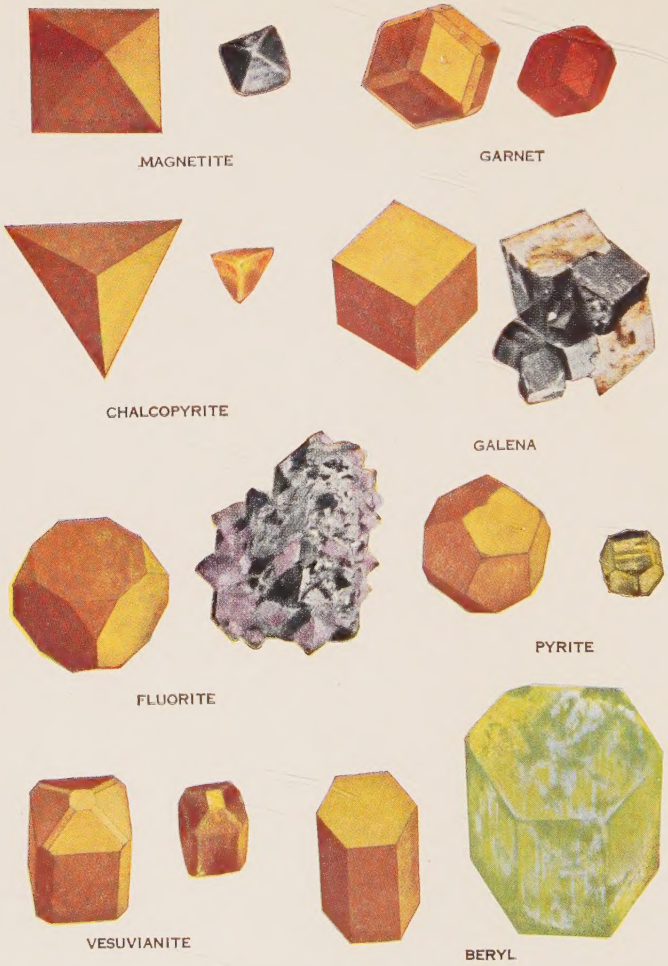

Courtesy American Museum of Natural History

SOME POLYHEDRAL CRYSTALS

The groups in the illustration given above show the correspondence between eight natural crystals and the wooden models of their equivalent ideal polyhedral forms on their left. "Crystallization is a curious and wonderful operation of Nature's geometry." The study of crystallography involves the application of numerous facts of geometry in connection with planes of symmetry and axes of twofold, threefold, fourfold, and sixfold symmetry. On a purely mathematical basis, the possibility of the existence of only thirty-two types of crystal symmetry has been deduced. There are, however, over 26,000 recorded drawings of crystals of minerals alone.

SOLID GEOMETRY

MODERN TEXT FOR SCHOOLS AND COLLEGES

BY

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AND

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Solid Geometry P-1-31

PREFACE

The objectives to be attained by a pupil in the study of solid geometry are (1) to develop further his spatial imagination, (2) to acquire skill in the representation of three dimensional figures, (3) to establish additional geometric facts concerning points, lines, surfaces, and solids, and (4) to become familiar with and to apply these facts in the calculation of problems which arise in natural life. Care has been taken in the preparation of this text to assist the pupil to a successful attainment of each of these objectives.

The recommendations of the National Committee, of the College Entrance Examination Board, and of various states have been fully met in content and in spirit. The basal propositions of the National Committee and of the College Board are appropriately lettered **N** and **C** respectively. The subsidiary or auxiliary propositions of these committees are marked with **n** and **c** respectively.

By selecting propositions indicated by **N** and **n**, or by **C** and **c**, or by their combinations, courses in solid geometry of various kinds may be arranged which will satisfy different curricula. Sufficient material has been included in the text so that any class may be given a thorough and an extensive course, but if it seems desirable, a more limited course may be given without destroying the sequence of fundamental theorems. The text may also be adapted to different grades of ability of pupils in a single class. Not only is this text flexible to a marked degree, but it may be used without loss no matter what text has been previously studied.

Attention is directed to certain other features which are briefly stated. The language used throughout is simple and exact. Extraneous matter has been limited to such material

as may add to the interest of the subject. The Introduction is brief and helpful. The definitions are precise and concise and are not redundant. The proofs are rigorous and for the most part are presented with sufficient fullness to enable the average pupil and the absent pupil to acquire a correct proof without assistance from some other source. Emphasis has been placed upon the important facts and not upon all. Numerous numerical exercises can be done mentally; they illustrate geometric facts without long numerical calculations. Many other exercises found in the named groups throughout the text and in the more difficult Miscellaneous Exercises at the close of the chapters offer excellent reviews of arithmetic and algebraic transformations and calculations. The applications found in the text and on p. 139 of the Supplement are varied and interesting. The Recreations (p. 149) are instructional. The Summary of Plane Geometry (p. 155) lists for easy reference and review the more important definitions, facts, and formulas of the previous course. The reasonable and rigorous treatment of limits by the use of substitution is satisfying to pupils. The General Miscellaneous Exercises (p. 137), adapted from those of the C. E. E. B., offer a newer and better kind of training in spatical imagination combined with computational skills. The Review Oral Exercises at the close of each chapter are useful for diagnostic and remedial work. The physical aspects of the book are attractive in the open paging, the clear type, and the excellence of the figures.

The authors make grateful acknowledgment for invaluable assistance in the preparation of the text to numerous teachers and especially to Dr. Randolph F. Clark, De Witt Clinton High School, New York City, and to Mr. Nelson L. Rorary, William L. Dickinson High School, Jersey City, N. J.

William W. Strader
Lawrence D. Rhoads

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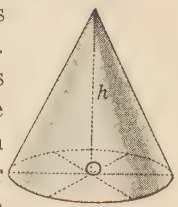
SOLID GEOMETRY

INTRODUCTION

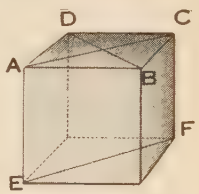
Why You Study Solid Geometry. The study of solid geometry is important and practical, since we live in a world of three dimensions. We move about on the earth, which is nearly spherical; we live in rooms, which are prismatic; we draw water from cylindrical pipes; we travel in vehicles which are formed of cylinders, spheres, and various rectangular solids. Our buildings, our furniture, our ornaments, our utensils, and other familiar objects depend upon the properties of certain standard geometric figures of three dimensions. A knowledge of the relationships of parts of such figures, of their surface areas, and of their volumes finds many uses in everyday life.

There are three principal objectives which you will attain in solid geometry. You will learn how to visualize and represent certain three-dimensional figures; you will learn how to prove certain Facts about these figures; you will learn how to apply these purely geometric truths to physical objects by performing numerous computations.

Applying Facts of Plane Geometry to Solid Geometry. In plane geometry, a figure is assumed or defined as lying in a plane, while in solid geometry all parts of a figure do not always lie in a single plane. Because of this difference, certain statements of plane geometry may not be taken as true in solid geometry. The statement, "Through a point in a line only one perpendicular can be drawn to the line," is true for plane



geometry, but not for solid geometry. For example, in the previous figure of a right circular cone, all the diameters of the base are perpendicular to the altitude at the point O. In such a figure, there is more than one perpendicular to the line at the given point, because the line and its perpendiculars do not all lie in one plane.



Another statement which we have learned in plane geometry is, "Two lines which do not meet are parallel." In solid geometry, such lines may or may not be parallel as may be seen in the annexed figure, in which AC and EF are parallel and BD and EF are not parallel and yet the lines in either pair do not meet.

There are three classes of Facts of plane geometry which may be used freely and without restriction in solid geometry. They are:

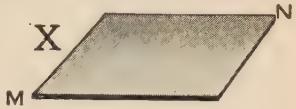
(1) statements which are true without referring to any plane, such as, the Principles concerning equal and unequal quantities and the majority of the definitions;

(2) statements in which it is evident that the figures may lie in one or several planes, such as, the theorems of congruence, similarity, and equality of polygons;

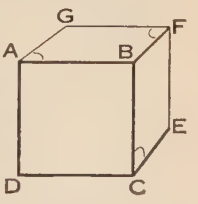
(3) statements in which the conditions stated determine a single plane in which all the parts must lie, such as, the theorems concerning a single triangle or circle and those dealing with two parallels cut by a transversal.

All other Facts of plane geometry may be used in solid geometry only after it has been shown that all the points and lines referred to lie in one plane. A great deal of attention is devoted to such cases in the beginning chapter of solid geometry.

Representation of a Plane. A plane is unlimited in extent in two dimensions and consequently it is impossible to represent completely such an unbounded figure. It is convenient, however, to represent a portion of a plane as a rectangle seen obliquely as shown in the annexed figure, which is called plane *X* or plane *MN*. In order to bring out the figure in correct relief, the front edge and one end may be made heavy and the surface may be slightly shaded as illustrated.



Representation of a Solid. In drawing a three-dimensional figure, it is impossible to represent all the parts in their true forms and relative sizes. All the faces of a cube, for example, are squares, all its edges are equal, and the face angles at the corners are right angles. When the figure of a cube is drawn, not more than two of its faces may appear as squares, the edges which extend from front to back are foreshortened, and many angles appear oblique. In drawing such figures, horizontal and vertical lines should be made respectively horizontal and vertical, while those lines which recede from the eye and which are in reality perpendicular to horizontal or vertical lines should be about one-half their true lengths and should meet horizontal lines at angles of about 45°.



Two Undefined Terms. Some terms used in mathematics are so elementary that it is difficult to give satisfactory definitions of them which employ more elementary and better known terms than those which are to be defined. This is true especially of such terms as "straight line" and "plane." We are so familiar, however, with these two concepts that it is not difficult to understand some of their properties and the relative positions which they may occupy.

Positions of Two Lines. We know that two straight lines in a plane occupy one of three relative positions:

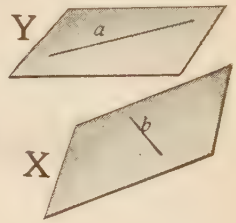
a) they may be parallel (when they have no point in common);

b) they may intersect (when they have one point in common;

c) they may coincide (when they have all points in common).

Besides these three relative positions, two straight lines in space may also occupy a fourth:

d) they may be skew lines (when they lie in different planes, have no point in common, and yet are not parallel). In the annexed figure, *a* and *b* are skew lines.



Positions of a Line and a Plane. A line and a plane have one of three relative positions in space:

a) they may be parallel (when they have no point in common);

b) they may intersect (when they have one point in common, which is called the foot of the line);

c) all points of the line may lie in the plane (when the line is said to lie in the plane and the plane is said to pass through the line).

Positions of Two Planes. Two planes occupy one of three relative positions in space:

a) they may be parallel (when they have no point in common);

b) they may intersect (when they have at least one point in common).

c) they may coincide (when they have all points in common or when they are determined by identical conditions).

Sketching Exercises

1. Make a sketch of a cube and letter its vertices. Name three sets of parallel lines. Name five pairs of perpendicular lines.

2. Make a sketch of a cylindrical tin can. The bases of your sketch are what figures? Draw a pair of parallel diameters, one in each base.

3. What is the locus of all points in space which are three inches from a given line segment? Draw a sketch to illustrate your answer.

4. What is the locus of points in space which are three inches from a given plane? Draw a sketch to illustrate your answer.

5. Make a sketch of a line and a plane which are parallel.

6. Make a sketch of a line and a plane which are perpendicular.

7. Make a sketch of a line and a plane which intersect.

8. Make a sketch of a plane which passes through a line.

9. Make a sketch of two intersecting planes.

10. Make a sketch of the figure of Proposition 2, p. 10.

11. Make a sketch of the figure of Proposition 5, p. 13.

12. Make a sketch of the figure of Proposition 7, p. 16.

13. Make a sketch of the figure of Proposition 13, p. 22.

Oral Exercises

1. State a Principle relating to perpendicular lines which is always true for plane geometry and which may not be true for solid geometry.

2. How may it be shown that a flagpole is perpendicular to a level playground by means of three guy-wires of equal length?

3. Must two lines be either perpendicular, parallel, or intersecting lines?

4. Describe in words the position of the intersections of the four diagonals of a cube.

5. If two planes are perpendicular, describe the relative positions of two lines, one in each plane, which cannot be perpendicular. Describe the relative positions of two lines which are perpendicular.

6. In the sentence "Two points determine a line," what is the meaning of the word *determine?*

7. If two points of a line lie in a plane, what can be said of the line and the plane?

8. What figure is formed by rotating an equilateral triangle about an altitude as an axis?

9. What figure is formed by rotating a square about a diagonal as an axis? about a side as an axis?

Courtesy The Bullinger Co.

PERPENDICULARS AND PARALLELS

The illustration shows numerous lines, planes, and lines and planes which are either perpendicular or parallel.

CHAPTER ONE

LINES AND PLANES IN SPACE

1. Principle. *A straight line lies in a plane, if two of its points lie in the plane.*

2. Corollary. *A straight line and a plane can intersect in only one point.*

3. Principle. *A plane can rotate about any line in it as an axis; and as the plane rotates, it can contain any external point in one and only one position.*

4. Corollary 1. *Any number of planes can pass through a given line.*

5. Corollary 2. *One plane and only one is determined by*

a) a straight line and an external point;

b) three points not in a straight line;

c) two intersecting straight lines;

d) two parallel lines;

e) any figure determined by a, b, c, or d.

6. Principle. *Two planes with one point in common have at least two points in common.*

7. Intersections. The intersection of a line and a surface or of two surfaces consists of all points common to them and no other points.

Oral Exercises

1. How many straight lines may pass through a point in a plane? two points in a plane?

2. Through how many points in a plane must a straight line pass so that the line may lie in the plane?

3. What is the difference in meaning between a plane intersecting a line and a plane passing through a line?

4. If two planes intersect, have they one point in common? two points?

5. How many planes may pass through a line? May any two of them be parallel? May two or more of them be perpendicular?

6. How many points determine a plane? What plane figure is determined by three points not in a straight line? Does this figure lie entirely in the plane determined by the three points?

7. Does a triangle determine a plane? Does the point of concurrence of the medians of a triangle lie in the same plane as the three vertices?

8. Why does a stool with three legs always rest firmly on a floor where a chair may be unsteady?

9. If two parallel lines are cut by a transversal, does the entire figure lie in one plane?

10. Why do all lines drawn to line AB from an external point P lie in one plane?

11. If two congruent rectangles lie in different planes and have their respective sides parallel, how many different planes are determined by the parallel sides taken in all possible combinations?

12. If a smooth sheet of paper is folded in a sharp crease, is the crease always a straight line?

13. What is meant by the intersection of two surfaces?

14. How many points determine a line? Is a curved or a broken line determined by two points?

15. How many planes are determined by a straight line and an external point?

16. If two planes have a straight line in common, may they coincide?

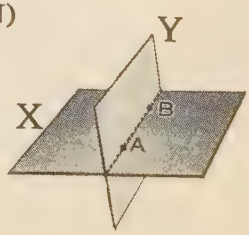
NOTE: The letters (**C**, **N**, **c**, and **n**) attached to the propositions in this text have the same significance as they had for Plane Geometry. The **C** and **N** indicate respectively the major propositions of the syllabi of the College Entrance Examination Board and of the National Committee; those propositions marked with **c** and **n** are the auxiliary and subsidiary propositions of the respective syllabi.

Proposition 1 (c N)

8. Theorem. *The intersection of two planes is a straight line.*

Given: Intersecting planes X and Y with one common point (A).

To prove: X and Y intersect in a straight line.



Proof: STATEMENTS	FACTS
1. Let A and B be any two points common to X and Y.	1. Given, § 6.
2. Draw straight line AB.	2. Why?
3. AB lies in X and in Y.	3. § 1.
4. Any point external to AB cannot lie in X and in Y.	4. § 5a.
5. X and Y intersect in the straight line AB.	5. § 7.

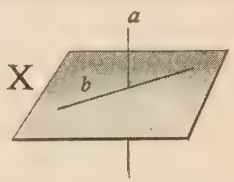
9. Collinear. Points or lines which lie in the same line are called *collinear points* or *lines.*

10. Coplanar. Points, lines, or planes which lie in the same plane are called *coplanar points, lines,* or *planes.*

11. Principle. *Any number of lines can be perpendicular to a line in space at any of its points.*

12. Perpendicular to a Plane. If a line meets a plane so that every line through its foot and lying in the plane is perpendicular to it, the line and the plane are said to be perpendicular to each other. Lines which are perpendicular to a plane are called *perpendiculars.*

NOTE: The attention of the pupil is called to the reversible nature of a definition. The definition of a line and a plane which are perpendicular is seldom used in its direct sense, but the definition in its reversed sense is often employed. For example, *if a line and a plane are known to be perpendicular,* any line in the plane which passes through the intersection of the line and the plane, is perpendicular to the given line. In the annexed figure, if $a \perp X$ and b lies in X, then by § 12 taken in its reversed sense, we know that $a \perp b$.



Oral Exercises

1. Are two perpendicular lines coplanar? In what kind of angle are the sides collinear?

2. Are the diagonals of a square and the mid-points of its sides coplanar?

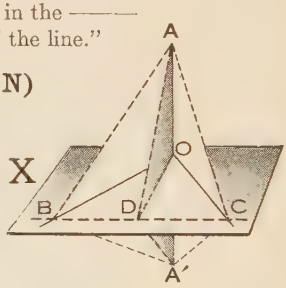
3. If three lines a, b, and c are each perpendicular to line d at point O, are a, b, and c coplanar? How many planes are determined by these four lines?

4. In plane geometry, are all the points collinear which are equidistant from two given points? In solid geometry are they collinear, coplanar, or neither?

5. If a line AO is perpendicular to a plane X at the point O, how many lines in X which pass through O may be perpendicular to AO?

6. What must be proved to show that a line is perpendicular to a plane?

7. Complete the statement, "Any point in the ——— of a line is equidistant from the ——— of the line."



Proposition 2 (C N)

13. Theorem. *A line perpendicular to each of two intersecting lines at their point of intersection is perpendicular to their plane.*

Given: $AO \perp OB$, $AO \perp OC$, and X, the plane of OB and OC.

To prove: $AO \perp X$.

Proof:

Statements	Facts
1. In X, draw OD, *any other line* through O. Draw BDC intersecting OB, OD, and OC in B, D, and C respectively.	1. Why?
2. Produce AO to A', making $OA' = OA$.	2. Why?
3. Connect A and A' with B, C, and D.	3. Why?
4. BO and CO are perpendicular bisectors of AA'.	4. Why?
5. $AB = A'B$, and $AC = A'C$.	5. Why?

The pupil may complete the proof by showing: $\triangle ABC \cong \triangle A'BC$; $\triangle ABD \cong \triangle A'BD$; $DA = DA'$ and $OA = OA'$; $DO \perp AA'$ or $AO \perp OD$; $AO \perp X$ (§ 12).

Note: It was stated previously that the definition of § 12 was seldom used in its direct sense. The reason for this may now be understood. Proposition 2 makes it possible to show that a line is perpendicular to a plane, *if it is perpendicular to two intersecting lines* which pass through its foot and which lie in the plane.

14. Corollary 1 (c). *Through a given point in a line, one plane and only one can be passed perpendicular to that line.*

HINT: In Fig. 1, draw CP and DP each $\perp AB$ at P, then $X \perp AB$ (§ 13).

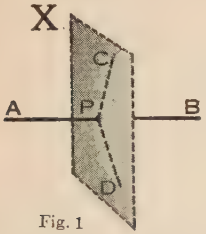


Fig. 1

In Fig. 2, suppose X and Y are both $\perp AB$ at P. Let Z pass through AB. Then $QP \perp AB$ and $RP \perp AB$, which is impossible.

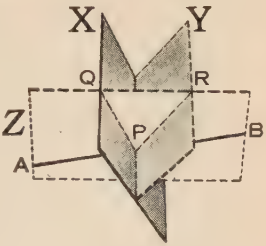


Fig. 2

15. Corollary 2 (c). *Through a point not in a given line, one plane and only one can be passed perpendicular to the line.*

HINT: In Fig. 1, suppose C is the external point to AB. Draw $CP \perp AB$ and draw another perpendicular to AB at P which does not lie in the plane of AB and CP.

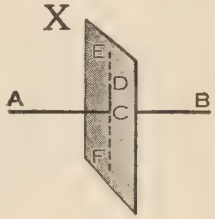
In Fig. 2, suppose QR the line and P the external point with X and Y each $\perp QR$, then $PQ \perp QR$ and $PR \perp QR$ which is impossible.

Proposition 3 (c N)

16. Theorem. *All the perpendiculars to a given line at a given point lie in the plane which is perpendicular to the given line at that point.*

Given: $X \perp AB$ at C, DC (any line) $\perp AB$ at C.

To prove: DC lies in X.



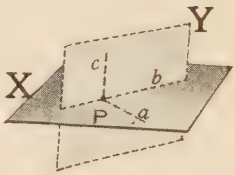
Proof:

STATEMENTS	FACTS
1. AB and DC determines a plane which intersects X in EF.	1. §§ 5c, 8.
2. $X \perp AB$.	2. Why?
3. $EF \perp AB$.	3. Why?
4. $DC \perp AB$.	4. Why?
5. EF and DC coincide.	5. Why?
6. Since EF lies in X, DC lies in X.	6. Statement 1.

17. Corollary (c n). *The locus of points equidistant from two given points is the plane which is the perpendicular bisector of the line joining them.*

Proposition 4 (C N)

18. Theorem. *Through a given point in a plane, one line and only one can be drawn perpendicular to that plane.*

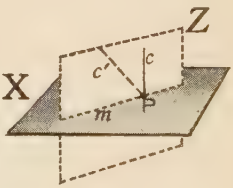


Given (1): Point P in X.

To prove (1): Through P, a line (c) can be drawn $\perp X$.

Proof (1):

Statements	Facts
1. In X, draw a (any line) through P.	1. Why?
2. Pass $Y \perp a$ through P.	2. Why?
3. X and Y intersect in b.	3. Why?
4. In Y, draw $c \perp b$ at P.	4. Why?
5. $c \perp a$.	5. Why?
6. $c \perp X$.	6. § 13.



Given (2): $c \perp X$ through P.

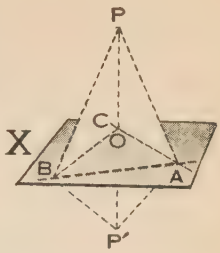
To prove (2): Any other line (c') through P is not $\perp X$.

Proof (2):

Statements	Facts
1. If c and c' are each $\perp X$ through P, they determine Z.	1. Why?
2. Z and X intersect in m.	2. Why?
3. $c \perp m$ and $c' \perp m$ at P.	3. Why?
4. This is impossible and c' is not $\perp X$.	4. Why?

Proposition 5 (C N)

19. Theorem. *Through a given point not in a plane, one line and only one can be drawn perpendicular to that plane.*

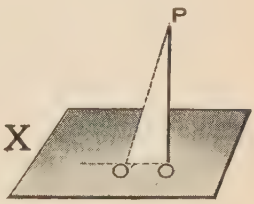


Given (1): Point P not in X.

To prove (1): Through P, a line (PO) can be drawn $\perp X$.

Proof (1):	STATEMENTS	FACTS
1.	From P, draw $PA \perp BA$ (any line) in X.	1. Why?
2.	In X, draw $CA \perp BA$.	2. Why?
3.	In the plane of CA and P, draw $PO \perp CA$.	3. Why?
4.	From O, draw OB to any point B in AB.	4. Why?
5.	Produce PO to P' making $OP' = PO$.	5. Why?
6.	Draw PB, $P'B$, and $P'A$.	6. Why?

The pupil may complete the proof by showing: $BA \perp P'A$; rt. $\triangle PAB \cong$ rt. $\triangle P'AB$; $BP = BP'$; $BO \perp POP'$; $PO \perp X$.



Given (2): $PO \perp X$.

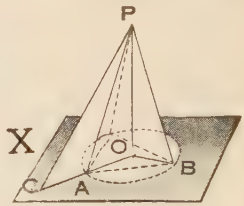
To prove (2): Any other line (PO') from P is not $\perp X$.

Proof (2):	STATEMENTS	FACTS
1.	Suppose PO' and PO are both $\perp X$ and that their plane cuts X in OO'.	1. §§ 5*c*, 8.
2.	$PO \perp OO'$ and $PO' \perp OO'$.	2. § 12.
3.	This is impossible and PO' is not $\perp X$.	3. Why?

20. Oblique to a Plane. If a line and a plane which intersect are not perpendicular, they are oblique to each other. Lines which are oblique to a plane are called *obliques*.

Proposition 6 (c n)

21. Theorem. *Obliques from any point in a perpendicular to a plane meeting the plane at equal distances from the foot of the perpendicular are equal, and conversely.*



Given: $PO \perp X$, $OB = OA$.

To prove: $PB = PA$.

(The pupil may complete the demonstration.)

22. Corollary 1 (c n). *Of two obliques from a point in a perpendicular to a plane, meeting the plane at unequal distances from the foot of the perpendicular, the one corresponding to the greater distance is the greater, and conversely.*

Given: $PO \perp X$, $OC > OB$.

To prove: $PC > PB$.

(The pupil may complete the demonstration.)

23. Corollary 2 (c n). *The perpendicular is the shortest line segment from a point to a plane, and conversely.*

(The pupil may complete the demonstration.)

24. Corollary 3 (c). *The locus of points equidistant from the vertices of a triangle is the line through the center of the circumscribed circle, perpendicular to the plane of the triangle.*

(The pupil may complete the demonstration.)

Oral Exercises

1. What is the foot of a perpendicular?

2. What is the meaning of the term oblique?

3. Discuss the derivation of the words locus, plane, perpendicular, intersection, and oblique.

4. Complete the statement, "All the perpendiculars to a given line at a given point lie in a plane ———."

5. Are three parallel lines necessarily coplanar? Why?

Exercises. Perpendicular Lines and Planes

1. Explain how to construct a line perpendicular to a plane from an external point.

2. What relation must exist between two lines so that a plane may be passed through one of them and be perpendicular to the other? Explain the construction.

3. How many planes are determined by four concurrent lines, no three of which are coplanar?

4. Two boys tested the perpendicularity of a flagpole in their school yard by stretching taut the rope tied to the top of the pole and finding how far from the foot of the pole the end of the rope met the ground at different points. What theorem of geometry were they applying? What conditions were necessary, if the pole was perpendicular?

5. If a, b, and c are three coplanar lines which are concurrent at O with a fourth line d perpendicular to a and to c, is $d \perp b$? Why? If d were given perpendicular to a alone, would d be perpendicular to b and to c?

6. If three lines are perpendicular to one another at a common point, any one of the lines is perpendicular to the plane of the other two.

7. Three non-collinear points equidistant from the extremities of a given line determine a plane which is the perpendicular bisector of the given line.

8. If a plane is perpendicular to a line at its mid-point, any point in the plane is equidistant from the extremities of the line.

9. $BO \perp AO$ and $CO \perp AO$. Complete the following: AO is ——— to the plane of ——— and ———, because "a line perpendicular ———."

10. What is the locus of points equidistant from all points of a circle? Prove your answer.

11. Explain how a boy may find out whether a center tent pole is perpendicular to the ground by making several applications of a steel square.

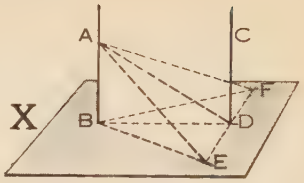
12. State the theorems of plane geometry which correspond to three theorems or corollaries of solid geometry you have studied thus far.

13. Tell why the following statement is either true, false, or indeterminate: A line perpendicular to each of two given lines is perpendicular to the plane of those lines.

14. Plane X is perpendicular to a at C and $DC \perp a$ at C, then DC lies in X. Quote the theorem which says this is true.

Proposition 7 (C N)

25. Theorem. *Two lines perpendicular to the same plane are parallel.*



Given: $AB \perp X$, $CD \perp X$.

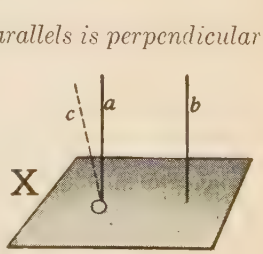
To prove: $AB \parallel CD$.

Proof: STATEMENTS	FACTS
1. Draw BD.	1. Why?
2. In X, draw $EF \perp BD$, making $ED = DF$.	2. Why?
3. Draw AD, and connect E and F with A and B.	3. Why?
4. $BF = BE$.	4. C 10*.
5. $AB \perp X$.	5. Why?
6. $AE = AF$.	6. § 21.
7. AD, BD, and CD are each $\perp EF$ at D.	7. E 11*, Cons., § 12.
8. AD, BD, and CD are coplanar.	§§ 16, 10.
9. AB and CD are coplanar.	9. § 1.
10. $AB \perp BD$, $CD \perp BD$.	10. § 12.
11. $AB \parallel CD$.	11. Why?

*NOTE: In Facts 4 and 7 references to the Summary of Plane Geometry on pp. 158-169 are designated by capital letters (C 10 and E 11). This scheme is used to indicate Facts of plane geometry in subsequent proofs.

26. Corollary 1 (c). *If one of two parallels is perpendicular to a plane, the other is also.*

HINT: Assume $c \perp X$ at O and show that c and a coincide.



27. Corollary 2 (c). *Two lines parallel to a third line are parallel to each other.*

HINT: Assume a plane (X) perpendicular to the third line.

28. Theorem. *If from the foot of a perpendicular to a plane, a straight line is drawn at right angles to any given line in the plane, the line from their intersection to any point in the perpendicular is perpendicular to the given line in the plane.*

HINT: Draw a perpendicular to a plane. Draw any line in the plane. From the foot of the perpendicular draw a line perpendicular to the line in the plane. Join their intersection to any point of the given perpendicular.

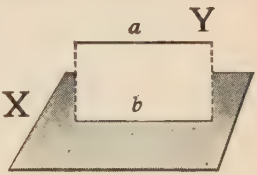
29. Parallel to a Plane. If a straight line and a plane have no point in common, they are said to be parallel to each other.

30. Parallel Planes. If two planes have no point in common, they are called *parallel planes.*

31. Skew lines. Two non-coplanar lines are called *skew lines.*

Proposition 8 (c N)

32. Theorem. *Any plane, which contains only one of two parallel lines, is parallel to the other line.*



Given: $a \parallel b$, X contains b but not a.

To prove: $X \parallel a$.

Proof:

STATEMENTS	FACTS
1. $a \parallel b$.	1. Why?
2. a and b determine Y.	2. Why?
3. X and Y intersect in b	3. Why?
4. If a (which lies entirely in Y) were to meet X, the intersection would be in b (which contains all points common to X and Y).	4. § 7.
5. $a \parallel b$ and they cannot meet.	5. Given, Def.
6. $X \parallel a$.	6. § 29.

33. Corollary 1. *A line parallel to a given plane is parallel to the intersection of the given plane with any plane through the line.*

HINT: If the given line and the intersection were to meet, the line would intersect the given plane.

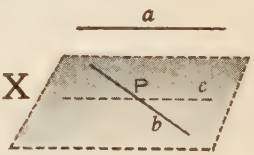
34. Corollary 2. *If each of two intersecting planes is parallel to a given line, the intersection of the planes is parallel to that line.*

HINT: Prove by the Indirect Method.

35. Corollary 3 (c). *Through either of two skew lines one plane and only one can be passed parallel to the other line.*

Given: a and b are skew lines.

To prove: A plane (X) and only one can be passed through b parallel to a.



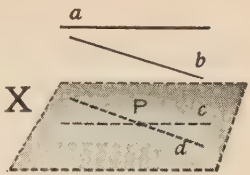
Proof:

STATEMENTS	FACTS
1. Through any point (P) in b, draw $c \parallel a$.	1. Why?
2. b and c determine a plane (X) parallel to a.	2. §§ 5c, 32.
3. If another plane (X') were to pass through b and be parallel to a, X' and X would coincide and form one plane; or, their intersection (b) would be parallel to a and would, therefore, be coplanar with a, which is impossible.	3. §§ 8, 34, Given.

36. Corollary 4 (c). *Through a given point not in either of two skew lines, one plane and only one can be passed parallel to each of the lines.*

Given: a and b are skew lines and P is any point not in a or in b.

To prove: A plane (X) and only one can be passed through P parallel to a and to b.



Proof:

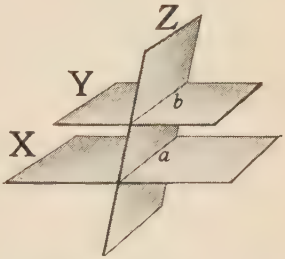
Statements	Facts
1. Through P draw $c \parallel a$ and $d \parallel b$.	1. Why?
2. c and d determine a plane (X) parallel to a and to b.	2. Why?
3. If another plane (X') were to pass through P and be parallel to a and to b, X and X' would coincide and form one plane or their intersection (call it m) would be parallel to a and b. Then a and b would be parallel, which is impossible.	3. Why?

Proposition 9 (c N)

37. Theorem. *The intersections of two parallel planes with a third plane are parallel lines.*

Given: $X \parallel Y$, a and b are the intersections of X and Y with Z.

To prove: $a \parallel b$.



Hint: a and b cannot meet, since $X \parallel Y$. Why? a and b lie in plane Z. Why? $a \parallel b$. Why?

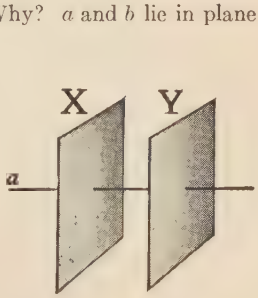
Proposition 10 (c N)

38. Theorem. *Two planes perpendicular to the same line are parallel.*

Given: $X \perp a$ and $Y \perp a$.

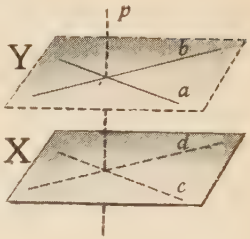
To prove: $X \parallel Y$.

Hint: If X and Y intersect, § 14 or § 15 is contradicted.



Proposition 11 (c)

39. Theorem. *Two intersecting lines, each of which is parallel to a given plane, determine a plane parallel to the given plane.*



Given: $a \parallel X$, $b \parallel X$, Y the plane of a and b.

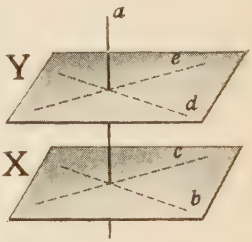
To prove: $Y \parallel X$.

Proof:

Statements	Facts
1. Through the intersection of a and b, draw $p \perp X$.	1. § 19.
2. Lines a and p and lines b and p determine planes which intersect X in c and d.	2. §§ 5*c*, 8.
3. $p \perp c$, $p \perp d$.	3. Why?
4. $a \parallel X$, $b \parallel X$.	4. Why?
5. $a \parallel c$, $b \parallel d$.	5. § 33.
6. $p \perp a$, $p \perp b$, $p \perp Y$.	6. Why?
7. $Y \parallel X$.	7. § 38.

Proposition 12 (c)

40. Theorem. *A line perpendicular to one of two parallel planes is perpendicular to the other plane.*



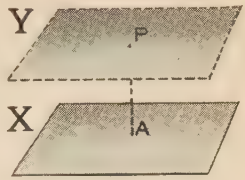
Given: $X \parallel Y$, $a \perp X$.

To prove: $a \perp Y$.

Proof:

Statements	Facts
1. Through a, pass two planes intersecting X in b and c and intersecting Y in d and e.	1. §§ 4, 8.
2. $a \perp X$.	2. Why?
3. $a \perp b$, $a \perp c$.	3. Why?
4. $X \parallel Y$, $e \parallel c$, $d \parallel b$.	4. Why?
5. $a \perp e$, $a \perp d$, $a \perp Y$.	5. Why?

41. Corollary (C). *Through a given point not in a given plane, one plane and only one can be passed parallel to that plane.*



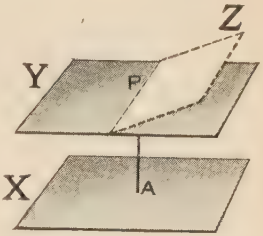
Given (1): Point P not in X.

To prove (1): Through P, a plane (Y) can be passed $\parallel X$.

Proof (1):	STATEMENTS	FACTS
	1. Draw $PA \perp X$.	1. § 19.
	2. Through P, pass $Y \perp PA$.	2. § 14.
	3. $X \parallel Y$.	3. § 38.

Given (2): $X \parallel Y$, $PA \perp X$, $PA \perp Y$.

To prove (2): Any other plane (Z) through P is not parallel to X.



HINT: If Z and Y are each parallel to X through P, $AP \perp Z$ and $AP \perp Y$, which is impossible by § 14.

NOTE: In proofs which consist of two parts, relations established in the first part may be used in the hypothesis or proof of the second part. Frequently two distinct figures are more clearly understood than a single composite figure. For this reason some proofs appear in two parts.

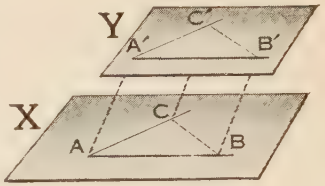
42. Distances. The length of the perpendicular segment from an external point to a plane is called the *distance from the point to the plane.* The length of a line segment perpendicular to one of two parallel planes and terminated by them is called the *distance between the planes.*

43. Corollary. *Two parallel planes are everywhere equidistant.*

44. Parallel in the Same Sense. Two parallel rays, which lie on the same side of a transversal through their end points, are said to be *parallel in the same sense.*

Proposition 13 (C N)

45. Theorem. *Angles, not in the same plane, whose sides are respectively parallel and in the same sense, are equal and their planes are parallel.*



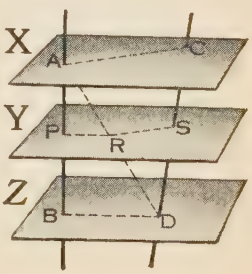
Given: $\angle A$ in X and $\angle A'$ in Y with their sides respectively parallel in the same sense.

To prove: $\angle A = \angle A'$, $X \parallel Y$.

Proof: STATEMENT	FACTS
1. Let $AB = A'B'$, $AC = A'C'$.	1. Why?
2. Draw BC, $B'C'$, AA', BB', and CC'.	2. Why?
3. $AC \parallel A'C'$, $AB \parallel A'B'$.	3. Why?
4. $AA'C'C$ and $AA'B'B$ are $\boxed{S}$.	4. Why?
5. $CC' = AA'$, $BB' = AA'$.	5. Why?
6. $CC' \parallel AA'$, $BB' \parallel AA'$.	6. Why?
7. $BB' = CC'$, $BB' \parallel CC'$.	7. Why?
8. $BB'C'C$ is a $\square$.	8. Why?
9. $BC = B'C'$.	9. Why?
10. $\triangle ABC \cong \triangle A'B'C'$.	10. Why?
11. $\angle A = \angle A'$.	11. Why?
12. $X \parallel A'B'$, $X \parallel A'C'$.	12. § 32.
13. $X \parallel Y$.	13. § 39.

Proposition 14 (c n)

46. Theorem. *Three parallel planes intercept proportional segments on any two transversals.*



Given: $X \parallel Y \parallel Z$, with transversals AB and CD determining points A, P, B and C, S, D respectively.

To prove: $AP : PB = CS : SD$.

Proof: STATEMENTS	FACTS
1. Draw AD.	1. Why?
2. AB and AD determine a plane which intersects the parallel planes in A, PR, and BD; CD and AD determine a plane which intersects the parallel planes in D, RS, and AC.	2. §§ 5*c*, 8.
3. $PR \parallel BD$ and $RS \parallel AC$.	3. § 37.
4. $\frac{AP}{PB} = \frac{AR}{RD}$ and $\frac{CS}{SD} = \frac{AR}{RD}$.	4. Why?
5. $\frac{AP}{PB} = \left(\frac{AR}{RD}\right) = \frac{CS}{SD}$.	5. Why?

Exercises. Parallel Lines and Planes

1. Explain how to pass a plane through a given line parallel to another given line.

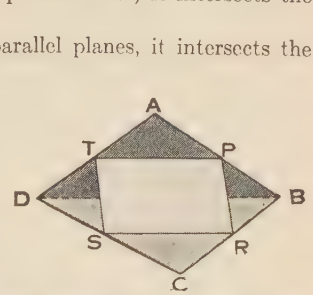
2. If three parallel planes cut off equal segments on one transversal they cut off equal segments on every transversal.

3. Prove two pairs of triangles similar in the figure of Proposition 14. (Two methods.)

4. If a plane intersects one of two parallel lines, it intersects the other line.

5. If a line intersects one of two parallel planes, it intersects the other plane.

6. If the vertices of $ABCD$ (a skew quadrilateral) are not all in the same plane, the lines joining in order the mid-points of its sides form a parallelogram.



7. The lines joining the mid-points of the opposite sides of any quadrilateral in space bisect each other.

8. The intersections of a plane with two parallel planes form angles which are either equal or supplementary.

9. If the distance between two parallel lines is 8 in., how far from their plane is the locus of points which are (*a*) 10 in. from the parallel lines? (*b*) x inches from the parallel lines?

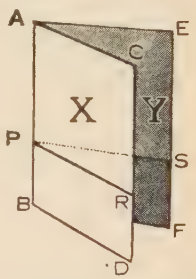
10. If a, b, and c are three parallel lines in space and the distances

between them are respectively 6 in., 8 in., and 10 in., how far from the three lines is the locus of points equidistant from them?

11. Name three Facts of solid geometry which you think are the most important of those already studied.

47. Half-Planes. Any straight line in a plane divides the plane into two parts called *half-planes*. The line is called the *edge* of either part.

48. Dihedral Angle. The figure formed by two half-planes which have a common edge is called a *dihedral angle*. The half-planes are called *faces* of the dihedral angle. The common edge is called the *edge* of the dihedral angle. The word *dihedral* means "two face." In the annexed figure, X and Y are faces and AB is the edge of the dihedral angle XY. A dihedral angle may be indicated in one of several ways: $\angle XY$, $\angle AB$, $\angle CBAF$, or $\angle X\text{–}AB\text{–}Y$.



49. Plane Angle of a Dihedral Angle. The angle formed by two lines, one in each face of a dihedral angle, perpendicular to its edge at the same point, is called a *plane angle of the dihedral angle*. In the figure of § 48, if $RP \perp AB$ and $SP \perp AB$, $\angle RPS$ is a plane angle of dihedral $\angle XY$. Since the point P may be any point in AB, the dihedral $\angle XY$ may have any number of plane angles. It is customary, however, to speak of any one of them as *the plane angle* of the dihedral angle.

50. Magnitude of a Dihedral Angle. Let us suppose that the two half-planes X and Y were originally coplanar on the same side of AB and that RP and SP were collinear on the same side of P. Any turning of X from Y about AB as an axis will produce a corresponding change of positions of RP and SP. If RP makes $\frac{n}{360}$ of one revolution from SP,

then X also makes $\frac{n}{360}$ of one revolution from Y. Because this is true, the following Principle may be stated:

51. Principle. *A dihedral angle equals numerically its plane angle.*

52. Corollary 1. *All plane angles of a dihedral angle are equal.*

53. Corollary 2. *Equal dihedral angles have equal plane angles, and conversely.*

54. Adjacent Dihedral Angles. Two dihedral angles which have a common edge and a common face between them are called *adjacent dihedral angles.*

55. Perpendicular Planes. If two equal adjacent dihedral angles are formed when one plane meets another, the planes are said to be perpendicular.

56. Right Dihedral Angles. A dihedral angle formed by two perpendicular planes is called a *right dihedral angle.*

Because this definition is reversible, there are two ways of proving two planes perpendicular: (1) by showing that they make right dihedral angles; (2) by showing that they form two equal adjacent dihedral angles. This is usually accomplished, in either case, by showing first that the corresponding relationship exists between their plane angles.

57. Vertical Dihedral Angles. Two non-adjacent dihedral angles formed by two intersecting planes are called *vertical dihedral angles.*

58. Other Kinds of Dihedral Angles. Dihedral angles are acute, obtuse, oblique, straight, complementary, supplementary, etc., if their plane angles possess like names.

59. Theorem. *The intersections of the faces of a dihedral angle with a plane perpendicular to its edge, form a plane angle of the dihedral angle.*

60. Corollary 1. *Vertical dihedral angles are equal.*

61. Corollary 2. *A dihedral angle can be bisected.*

Exercises. Dihedral Angles

1. If two dihedral angles are equal, their complementary dihedral angles are equal.

2. If three planes intersect in three parallel lines, the sum of the dihedral angles formed within the figure equals two right dihedral angles.

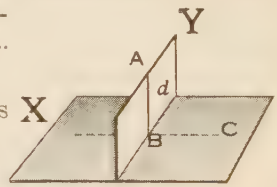
3. Alternate interior dihedral angles of parallel planes are equal.

4. Planes forming equal alternate interior dihedral angles with a transversal are parallel.

5. Dihedral angles whose faces are parallel (or perpendicular) each to each are either equal or supplementary.

Proposition 15 (C N)

62. Theorem. *If two planes are perpendicular to each other, any line in one plane, perpendicular to their intersection, is perpendicular to the other plane.*



Given: $X \perp Y$ in line d, AB lies in Y and is $\perp$ to d.

To prove: $AB \perp X$.

Proof:

Statements	Facts
1. In X, draw $CB \perp d$ at B.	1. Why?
2. $AB \perp d$.	2. Why?
3. $\angle ABC$ is the plane angle of dihedral $\angle YX$.	3. § 49.
4. Dihedral $\angle XY$ = rt. dihedral $\angle$.	4. Given, § 56.
5. $\angle ABC$ = rt. $\angle$.	5. § 51.
6. $AB \perp BC$.	6. Why?
7. $AB \perp X$.	7. Why?

63. Corollary 1 (c). *A perpendicular to one of two perpendicular planes, at any point of their intersection, lies in the other plane.*

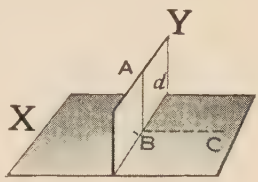
Hint: In the preceding figure, if d is the intersection of two perpendicular planes (X and Y) and $AB \perp d$ at B any point of X, any line in $Y \perp d$ at B coincides with AB (§ 18).

64. Corollary 2 (c). *A perpendicular to one of two perpendicular planes, from any point in the other, lies in that plane.*

HINT: In the figure of Proposition 15, if d is the intersection of two perpendicular planes (X and Y) and $AB \perp d$ from A in Y, a perpendicular from A to d coincides with AB (§ 19).

Proposition 16 (C N)

65. Theorem. *Any plane through a perpendicular to a given plane is also perpendicular to the given plane.*



Given: $AB \perp X$, Y is any plane passing through AB, intersecting X in d.

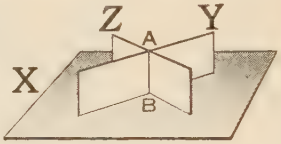
To prove: $Y \perp X$.

Proof: STATEMENTS	FACTS
1. In X, draw $CB \perp d$ at B.	1. Why?
2. $AB \perp X$.	2. Why?
3. $AB \perp d$ at B.	3. Why?
4. $\angle ABC$ is the plane angle of dihedral $\angle YX$.	4. Why?
5. $AB \perp BC$.	5. § 12.
6. $\angle ABC$ = rt. $\angle$.	6. Why?
7. Dihedral $\angle YX$ = rt. dihedral $\angle$.	7. § 51.
8. $Y \perp X$.	8. § 56.

66. Suggestion to Pupils. Since the statements of each of the last four sections (§§ 62 65) relate to a pair of perpendicular planes and a line which is perpendicular to one of them, they are more frequently confused than any other statements of solid geometry. Pupils should give special attention to the thorough understanding of the ideas expressed in these theorems. Fix in mind exactly what each statement means.

Proposition 17 (C N)

67. Theorem. *If each of two intersecting planes is perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Given: $Z \perp X$, $Y \perp X$, Z and Y intersect in AB.

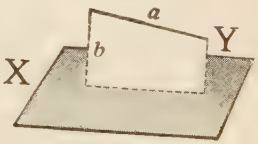
To prove: $AB \perp X$.

Proof:

Statements	Facts
1. Through B, the point common to X, Y and Z, draw a perpendicular to X.	1. § 18.
2. This perpendicular lies in Y and in Z.	2. § 63.
3. AB coincides with this perpendicular, that is, $AB \perp X$.	3. §§ 8, 18.

Proposition 18 (c)

68. Theorem. *Through a line, not perpendicular to a given plane, one plane and only one can be passed perpendicular to the given plane.*



Given: Line a not $\perp X$.

To prove: Through a, one plane (Y) and only one can be passed $\perp X$.

Proof:

Statements	Facts
1. From any point of a draw $b \perp X$.	1. § 19.
2. a and b determine Y.	2. Why?
3. $Y \perp X$.	3. § 65.
4. If there were two planes through a each $\perp X$, a would be $\perp X$; which is impossible.	4. § 67, Given.

69. Projections. The projection of a point upon a plane is the foot of the perpendicular from the point to the plane. The projection of any line (straight, curved, or broken)

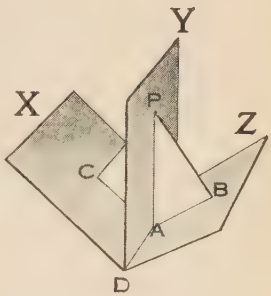
upon a plane is the locus of the projections of its points upon that plane.

70. Corollary. *The projection of a straight line, on a plane not perpendicular to it, is a straight line.*

HINT: In the preceding figure, pass Y through a and $\perp X$. The foot of a perpendicular to X from any point in a lies in the intersection of X and Y (§ 64).

Proposition 19

71. Theorem. *Any point in the plane bisecting a dihedral angle is equidistant from the faces of the dihedral angle, and conversely.*



Given: Dihedral $\angle XY$ = dihedral $\angle YZ$, P is any point in Y, $PC \perp X$, $PB \perp Z$.

To prove: $PC = PB$.

Proof:

STATEMENTS	FACTS
1. PC and PB determine a plane ($PCAB$) intersecting X, Y, and Z in CA, PA, and BA respectively.	1. §§ 5*c*, 8.
2. $PC \perp X$, $PB \perp Z$.	2. Why?
3. Plane $PCAB \perp X$ and $\perp Z$.	3. § 65.
4. $DA \perp$ plane $PCAB$.	4. § 67.
5. DA is perpendicular to CA, PA, and BA.	5. § 12.
6. $\angle PAC$ and $\angle PAB$ are plane angles of dihedral $\angle XY$ and dihedral $\angle YZ$ respectively.	6. Why?
7. Dihedral $\angle XY$ = dihedral $\angle YZ$.	7. Why?
8. $\angle PAC = \angle PAB$.	8. § 53.
9. $PC \perp CA$, $PB \perp AB$.	9. Why?
10. $PC = PB$.	10. C 9.

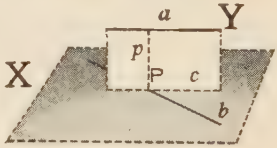
(The converse proof may be done by the pupil.)

72. Corollary (c n). *The locus of points equidistant from two intersecting planes is the pair of planes bisecting the dihedral angles formed by the given planes.*

HINT: Use the direct theorem and the converse of § 71. What is the corresponding theorem of plane geometry?

Proposition 20 (c n)

73. Theorem. *Between any two skew lines there is one common perpendicular and only one.*



Given (1): a and b are two skew lines.

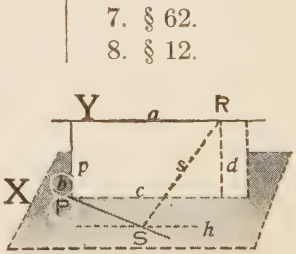
To prove (1): There is a common perpendicular (p) to a and to b.

Proof (1):

STATEMENTS	FACTS
1. Through b pass $X \parallel a$.	1. § 35.
2. Through a pass $Y \perp X$, intersecting X in c.	2. §§ 68, 8.
3. $c \parallel a$.	3. § 33.
4. If c were also parallel to b, then a would be parallel to b; but, a is not parallel to b, so a intersects b at some point P.	4. § 27, Given.
5. In Y, draw $p \perp c$ at P.	5. Why?
6. $p \perp a$.	6. E 1.
7. $p \perp X$.	7. § 62.
8. $p \perp b$.	8. § 12.

Given (2): a and b are two skew lines, $p \perp a$, $p \perp b$.

To prove (2): Any other line (s) joining a and b is not a common perpendicular to them.



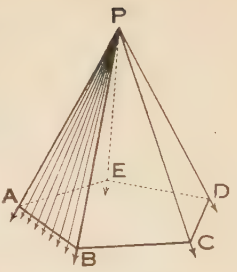
Proof (2): STATEMENTS	FACTS
1. Let X, Y, and c be determined as in Proof (1).	1. §§ 35, 68, 8.
2. Suppose $s \perp a$ at R and $s \perp b$ at S, and let the plane determined by a and S intersect X in h.	2. §§ 5a, 8.
3. $a \parallel h$.	3. § 33.
4. $s \perp h$.	4. E 1.
5. $s \perp X$.	5. § 13.
6. In Y, from R draw $d \perp c$.	6. Why?
7. $d \perp X$.	7. § 62.
8. s and d cannot both be perpendicular to X from R, so s is not $\perp b$.	8. § 19.

74. Inclinations of Lines to Planes. The inclination of a perpendicular to a plane is the right angle formed by the perpendicular and any line through its foot which lies in the plane. The inclination of an oblique to a plane is the acute angle formed by the oblique and its projection on the plane.

75. Polyhedral Angle. The figure generated by a moving ray that continuously interesects a fixed polygon as the end point of the ray remains fixed in a point non-coplanar with the polygon is called a *polyhedral angle.* If the polygon is convex, the figure is called a convex polyhedral angle.

76. Parts of a Polyhedral Angle. In a polyhedral angle, the fixed point is called the *vertex;* the moving ray in any of its positions is called an *element;* any element through a vertex of the polygon is called an *edge;* the portion of the plane determined by any two consecutive edges and included by them is called a *face;* the angle formed by any two consecutive edges is called a *face angle;* the dihedral angle formed by any two intersecting faces is called a *dihedral angle* of the *polyhedral angle.* If the fixed polygon has n

sides, the polyhedral angle has n edges, n faces, n face angles, and n dihedral angles.



In the annexed figure, P (or P–$ABCDEA$) is a polyhedral angle of five faces; PA, PB, PC, etc., are edges; $\angle APB$, $\angle BPC$, $\angle CPD$, etc., are face angles, and A–PB–C, B–PC–D, etc., are dihedral angles of the polyhedral angle.

77. Congruent Figures. Figures which can be made to coincide are called *congruent figures*.

78. Congruent and Symmetric Polyhedral Angles. Polyhedral angles with their face angles and their dihedral angles respectively equal and arranged in the same order are called *congruent polyhedral angles*. Polyhedral angles with their face angles and their dihedral angles respectively equal and arranged in reverse order are called *symmetric polyhedral angles*. Except in special cases, symmetric polyhedral angles cannot be made to coincide.

If the edges of a polyhedral angle are produced through the vertex, another polyhedral angle is formed. The two polyhedral angles are called vertical polyhedral angles. In the annexed figure, P–ABC and P–$A'B'C'$ are symmetric polyhedral angles as well as vertical polyhedral angles.

79. Special Kinds of Polyhedral Angles. Polyhedral angles of three, four, five, six, eight, or ten faces are called respectively *trihedral*, *tetrahedral*, *pentahedral*, *hexahedral*, *octahedral*, or *decahedral angles*. In our work we shall deal almost entirely with trihedral angles.

A trihedral angle with two equal face angles is called an *isosceles trihedral angle*.

Trihedral angles with one, two, or three right dihedral angles are called respectively *rectangular*, *birectangular*, or *trirectangular trihedral angles.*

80. Theorem A. *Trihedral angles with two face angles and the included dihedral angle of one equal to the respective parts of another and arranged in the same order are congruent.*

Theorem B. *Trihedral angles with two dihedral angles and the included face angle of one equal to the respective parts of another and arranged in the same order are congruent.*

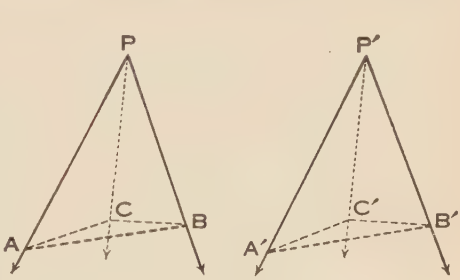
NOTE: These two theorems may be assumed without proof. However, if the teacher so desires, proofs may be given by the Method of Superposition.

Proposition 21 (c n)

81. Theorem. *Trihedral angles with their face angles respectively equal and arranged in the same order are congruent.*

Given: Trihedral angles P and P' with their face angles respectively equal and arranged in the same order.

To prove: Trihedral $\angle P \cong$ trihedral $\angle P'$.



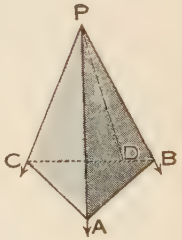
HINT: Take $PB = P'B'$; draw AB and $CB \perp PB$ at B; draw $A'B'$ and $C'B' \perp P'B'$ at B'; $\measuredangle ABC$ and $A'B'C'$ are respectively the plane angles of the dihedral $\measuredangle PB$ and $P'B'$; rt. $\triangle ABP \cong$ rt. $\triangle A'B'P'$ and rt. $\triangle CBP \cong$ rt. $\triangle C'B'P'$ (*l.a.* = *l.a.*); $PA = P'A'$ and $PC = P'C'$ (C 1); $\triangle CPA \cong \triangle C'P'A'$ (*s.a.s.* = *s.a.s.*); $\triangle ABC \cong \triangle A'B'C'$ (*s.s.s.* = *s.s.s.*); plane $\angle ABC$ = plane $\angle A'B'C'$ (G 1); dihedral $\angle PB$ = dihedral $\angle P'B'$ (§ 53); trihedral $\angle P \cong$ trihedral $\angle P'$ (§ 80*A*).

82. Corollary 1 (c n). *Trihedral angles with their face angles respectively equal and arranged in the reverse order are symmetric.*

83. Corollary 2. *If two trihedral angles have equal face angles, the dihedral angles of one are equal to the dihedral angles of the other.*

Proposition 22 (c n)

84. Theorem. *The sum of any two face angles of a convex trihedral angle is greater than the third face angle.*



Given: Trihedral $\angle P$ with $\angle BPC$, the greatest face angle.

To prove: $\angle APB + \angle APC > \angle BPC$.

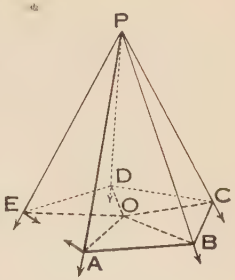
Proof:

Statements	Facts
1. In the face CPB, let $PD = PA$ and $\angle DPC = \angle APC$. Draw CDB, CA, and AB.	1. Why?
2. $PC = PC$.	2. Why?
3. $\triangle APC \cong \triangle DPC$.	3. *s.a.s.* = *s.a.s.*
4. $AC = CD$.	4. Why?
5. $AC + AB > CD + DB$.	5. Why?
6. $AB > DB$.	6. A 11.
7. $PB = PB$.	7. Why?
8. $\angle APB > \angle BPD$.	8. P 14.
9. $\angle APB + \angle APC > \angle BPD + \angle DPC$, or $\angle APB + \angle APC > \angle BPC$.	9. Why?

Question. What theorem of plane geometry corresponds most closely with the theorem of § 84?

Proposition 23 (c n)

85. Theorem. *The sum of the face angles of any convex polyhedral angle is less than 360°.*



Given: P is a convex polyhedral angle of n faces.

To prove: The sum of the face angles at P is less than 360°.

Proof:

Statements	Facts
1. Let $ABCD.....A$ be the fixed polygon of polyhedral $\angle P$. From any point (O) within $ABCD.....A$, draw OA, OB, OC, etc.	1. Why?
2. There are n triangles whose common vertex is P and n triangles whose common vertex is O. Denote the expression, "the sum of" by Σ, the $\triangle$ whose common vertex is P by the $P\ \triangle$, and the $\triangle$ whose comnon vertex is O by the $O\triangle$.	2. § 76.
3. $\Sigma\ \measuredangle$ of the $P\ \triangle = n\ (180°) = \Sigma\ \measuredangle$ of the $O\ \triangle$.	3. Why?
4. $\angle PBA + \angle PBC > \angle OBA + \angle OBC$, $\angle PCB + \angle PCD > \angle OCB + \angle OCD$, etc.	4. § 84.
5. Σ base $\measuredangle$ of the $P\ \triangle > \Sigma$ base $\measuredangle$ of the $O\ \triangle$.	5. Ax. of Add.
6. $\Sigma\ \measuredangle$ at $P < \Sigma\ \measuredangle$ at O.	6. A 14.
7. $\Sigma\ \measuredangle$ at $O = 360°$.	7. Why?
8. $\Sigma\ \measuredangle$ at $P < 360°$.	8. A 3.

Question. What theorem of plane geometry corresponds most closely with the theorem of 85?

Exercises. Polyhedral Angles

1. How many polyhedral angles are formed by three planes which intersect so that they have only one point in common?

2. If the fixed polygon of a polyhedral angle has five sides, how many faces has the polyhedral angle? Why?

3. Are equal polyhedral angles symmetric? Why?

4. Draw a sketch of a pair of vertical polyhedral angles which are symmetric and which can be made to coincide. If the fixed polygon is taken as a regular polygon, how is the point which is common to all the faces of the two polyhedral angles related to the center of the fixed polygon?

5. What kind of trihedral angles are formed at the corners of a rectangular box? Are such trihedral angles equal or symmetric? Why?

6. Can the face angles of a trihedral angle be 100°, 120°, 140°?

Exercises. Parallels and Perpendiculars

1. Two points, on the same side of a plane and equidistant from it, determine a line parallel to the plane.

2. Three points, on the same side of a plane and equidistant from it, determine a plane parallel to the given plane.

3. If two parallel planes intersect each of two other parallel planes, the four lines of intersection are parallel in pairs.

4. If a line and a plane are non-coplanar and each is perpendicular to the same line, they are parallel.

5. If a line is parallel to a plane, any plane perpendicular to the line is also perpendicular to the plane.

6. If two angles, not in the same plane, have their sides respectively parallel but not in the same sense, are they equal and are their planes parallel? Prove your answers.

7. If two planes are each parallel to a line, the intersections of a plane through the line with the given planes are parallel.

8. The projections of two parallel lines on a plane are parallel.

9. A line which is equal to its projection on a plane is parallel to the plane.

10. If two planes are perpendicular to each other, a line non-coplanar with either which is perpendicular to one plane is parallel to the other plane.

Exercises. Loci

1. What is the locus of points in space equidistant from a given plane?

2. What is the locus of points equidistant from two given planes and also equidistant from two given points?

3. What is the locus of the perpendiculars to a given line at a given point?

4. If the edge of the windshield of an automobile is parallel to the roadway, what is the geometric figure generated by such an edge as the car moves over a level road?

5. In a given plane, what is the locus of points equidistant from two points not in the plane?

6. In a given plane, what is the locus of points at a given distance from a point not in the plane?

7. What is the locus of points in space equidistant from two given parallel planes?

8. What is the locus of the mid-points of all line segments which have their extremities in two parallel planes?

9. What is the locus of points in space equidistant from the vertices of a given triangle? from the sides of the triangle?

10. What is the locus of points equidistant from three non-coplanar lines parallel in pairs?

11. A triangle revolves about one of its sides as an axis. What is the locus of the opposite vertex?

12. What is the locus of points in space equidistant from the vertices of given square? from the sides of the square?

13. What is the locus of points equidistant from the faces of a dihedral angle (*a*) and lying in a given plane; (*c*) and equidistant from two given points; (*c*) and equidistant from three given points; (*d*) and equidistant from two parallel lines. Discuss the impossible cases.

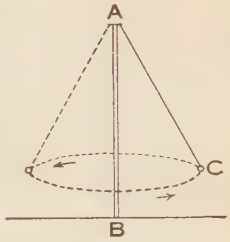
14. How far from the plane of a 10-in. square are points which are 13 in. from each of its sides? 20 in. from each of its vertices?

15. If points A and B are 16 in. apart, what is the locus of points 10 in. from A and B? Find the length of such locus.

16. If a point is 8 in. from a given plane, what is the locus of points which lie in the given plane and are 16 in. from the given point? Find the length of such locus.

17. If two planes intersect to form right dihedral angles, describe the locus of points 2 in. from one plane and 4 in. from the other plane. How far from the edge of the dihedral angle are all such points?

18. The annexed figure represents a "Giant Stride" often found on childrens' playgrounds. AB is a vertical iron pole. AC is a 10-ft. rope fastened to the pole at A by means of a movable ring which permits AC to revolve about the pole and at various angles. Find the length of the path generated by C, if the rope is taut as AC revolves and $\angle BAC$ is constantly equal to 30°; if the angle is constantly equal to 45°.



19. Where is a point 5 units from the vertices of a right triangle whose sides are 6, 8, and 10? Where is a point 7 units from each of the vertices of the same triangle? How far from the plane of the triangle is the latter point?

20. Right triangle with $\angle C = 90°$, $a = 6$, and $b = 8$ revolves about one of its sides as an axis. Find the radius of the locus generated by the opposite vertex, if the axis of revolution is in turn a; b; c.

21. A point 10 in. from each face of a dihedral angle is how far from its edge, if the plane angle of the dihedral angle is (*a*) 90°; (*b*) 60°; (*c*) 120°?

22. What is the locus of points 2 in. distant from the mid-point of a line segment 5 in. long and equally distant from the end points of the line segment. How far is it from any point of the locus to either end point?

Review Oral Exercises

1. State four conditions which determine a plane.

2. State three possible relative positions (*a*) for two coplanar lines; (*b*) for a line and a plane; (*c*) for two planes.

3. Through a given point how many lines can be drawn perpendicular (*a*) to a given line; (*b*) to a given plane?

4. Through a given point how many planes can be drawn perpendicular (*a*) to a given line; (*b*) to a given plane?

5. Through a given point how many lines can be drawn parallel (*a*) to a given line; (*b*) to a given plane?

6. Through a given point, how many planes can be drawn parallel (*a*) to a given line; (*b*) to a given plane?

7. What is the intersection of two lines? of a line and a plane? of two planes? of three planes?

8. State three conditions which make planes parallel.

9. State three conditions which make planes perpendicular.

10. State three conditions which make a line perpendicular (*a*) to a line; (*b*) to a plane.

11. State three conditions which make a line parallel (*a*) to a line; (*b*) to a plane.

12. Are planes necessarily parallel, if they contain parallel lines? Discuss the conditions.

13. Are lines necessarily parallel, if they lie in parallel planes? Discuss the conditions.

In the exercises (14–40) which follow, a, b, c represent straight lines, P a point, and X, Y, Z planes.

14. If $a \perp b$, is every plane containing a perpendicular to b?

15. If $a \parallel b$, is every plane containing b parallel to a?

16. If $X \perp Y$, is every line in X perpendicular to Y?

17. If $X \parallel Y$, is every line in Y parallel to X?

18. If $X \perp Y$, is every line perpendicular to X also perpendicular to Y?

19. If $X \parallel Y$, is every line parallel to Y also parallel to X?

20. If $a \perp X$, is every plane through a perpendicular to X?

21. If $a \parallel X$, is every plane through a parallel to X?

In Exercises 22–40, translate each symbolic statement into words. Does each verbal statement correspond to a previous proposition or corollary?

22. If $a \perp b$ and c, $a \perp$ plane of b and c.

23. If X contains a and $a \perp Y$, $X \perp Y$.

24. If a and $b \perp X$, $a \parallel b$.

25. If $a \parallel b$ and $X \perp a$, $X \perp b$.

26. If $a \parallel b$ and $a \parallel c$, $b \parallel c$.

27. If X contains a and $a \parallel b$, $X \parallel b$.

28. If X and $Y \perp a$, $X \parallel Y$.

29. If a and $b \parallel Y$, the plane of a and $b \parallel Y$.

30. If $a \perp X$ and $X \parallel Y$, $a \perp Y$.

31. If X and $Y \perp Z$ and X and Y intersect in a, $a \perp Z$.

32. If $X \parallel Y$ and Z intersects X and Y in a and b respectively, $a \parallel b$.

33. If $a \parallel c$ and $b \parallel d$ in the same sense, $\angle ab = \angle cd$.

34. X can be drawn through P and $\perp a$.

35. X can be drawn through P and $\parallel Y$.

36. X can be drawn through P and $\parallel a$ and b.

37. X can be drawn through a and $\perp Y$.

38. a can be drawn through P and $\perp X$.

39. a can be drawn perpendicular to b and c.

40. X can be drawn through a and $\parallel b$.

41. State five theorems or corollaries which are "required theorems" of the College Entrance Examination Board. (They are indicated in this text by *C*.)

42. State five theorems or corollaries which are "required theorems" of any local or state syllabus which you may have to meet. Do you know how to prove each of these theorems?

43. State three theorems of Chapter One which you think are most important. Why do you think so?

Miscellaneous Propositions

1. A straight line lies in a plane, if it has a point in the plane and is parallel to a line in the plane.

2. If a line is parallel to a plane, all points in the line are equidistant from the plane.

3. If a plane is perpendicular to each of two other perpendicular planes, the intersection of two of the planes is perpendicular to the third plane and also perpendicular to each of the other intersections.

4. If two parallel planes cut each of three or more concurrent lines, the corresponding segments are proportional.

5. If two planes intersect and pass through two parallel lines, one in each plane, these lines are parallel to the intersection of the planes.

6. The acute angle which a straight line makes with its projection on a plane is the least angle which it makes with any line in the plane.

7. A plane perpendicular to the edge of a dihedral angle is perpendicular to each face.

8. A line parallel to the edge of a dihedral angle is parallel to each face, and conversely.

9. If each of three planes intersects the other two, their intersections are collinear, parallel, or concurrent.

10. Segments of parallel lines cut off between parallel planes are equal.

11. Two dihedral angles have the same ratio as their plane angles.

12. If two planes are parallel to a third plane, the three planes are parallel.

13. If a straight line and a plane are parallel, any plane perpendicular to the line is also perpendicular to the given plane.

14. The common perpendicular to two skew lines is the shortest line between them.

15. A line lies in a plane, if it has a point in the plane and the line and the plane are both parallel to a second line (or plane).

16. Complete and prove: If a ——— intersects one of two parallel ———, it intersects the other also. (There are at least three such theorems.)

17. Through a given external point to a plane, draw a line parallel to the given plane. Explain your construction in full.

18. Through a given external point to a plane, pass a plane parallel to the given plane.

19. Through a given line, pass a plane parallel to a second given line.

20. Through a given point, pass a plane parallel to any two given straight lines.

21. Bisect a given dihedral angle. Explain your construction in full.

22. The projection of a line on a plane equals the length of the line multiplied by the cosine of the angle formed by the line and its projection.

23. The projection of a plane figure on a plane equals the area of the figure multiplied by the cosine of the angle which the plane makes with the plane of the figure.

CHAPTER TWO

POLYHEDRONS

86. Polyhedron. A geometric solid whose boundary consists of plane polygons is called a *polyhedron*. If every plane which cuts a polyhedron forms a convex polygon, the polyhedron is said to be *convex*.

NOTE: A polygon or a closed curve is convex, if a straight line can intersect the polygon or curve in not more than two points. Unless otherwise stated, all solids dealt with in this course are convex.

A boundary of a polyhedron or a plane boundary of any solid may be considered either as a closed line or as the portion of the plane determined by and bounded by the closed line, according to the context.

87. Parts of a Polyhedron. In a polyhedron, the sides of the bounding polygons are called *edges;* the intersections of the edges are called *vertices;* the portions of planes determined by and bounded by the polygons are called *faces;* the lines joining any two vertices not in the same face are called *diagonals;* the vertices, edges, and faces taken together form the *surface* of a polyhedron.

88. A polyhedron of { four, six, eight, twelve, twenty } faces is called { a *tetrahedron*, a *hexahedron*, an *octahedron*, a *dodecahedron*, an *icosahedron* }

89. Regular Polyhedron. A polyhedron whose faces are congruent regular polygons and whose polyhedral angles are equal is called a *regular polyhedron*. The figures on the following page are regular polyhedrons.

Oral Exercises

1. What is the least number of faces a polyhedral angle can have?

2. The sum of the face angles of a polyhedral angle is less than what number of degrees?

Tetrahedron Hexahedron Octahedron Dodecahedron Icosahedron

3. How many equilateral triangles can meet at a vertex to form a polyhedral angle?

4. What is the greatest number of regular polyhedrons possible having equilateral triangles as faces? What are their names?

5. How many squares can meet at a vertex to form a polyhedral angle?

6. How many polyhedrons can have squares as faces? How many faces has such a polyhedron? What is its name?

7. How many degrees are in each interior angle of a regular pentagon?

8. How many regular pentagons can meet at a vertex to form a polyhedral angle?

9. How many regular polyhedrons can have regular pentagons as faces? Name such polyhedrons and state the number of faces.

10. How many degrees are in each interior angle of a regular hexagon? a regular octagon?

11. Can the faces of a regular polyhedron be regular polygons of more than five sides? Why?

12. What is the greatest number of regular polyhedrons?

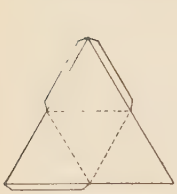
90. Theorem (c n). *The number of regular polyhedrons is limited to five.*

91. The Five Regular Solids. The theorem of § 90 does not state that there are five regular polyhedrons; it states that there cannot be more than five. It is possible, however, to prove their existence and to construct them. The five regular polyhedrons have been the subject of considerable study since early times. The Pythagoreans knew

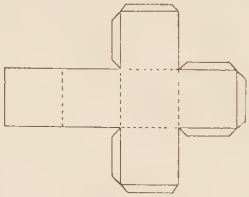
that there were five; Euclid proved that there cannot be more than five; and because Plato taught numerous important theorems concerning them and assigned rather mystical properties to them in his study of philosophy, they are often called the *Platonic Bodies*.

Models of the regular polyhedrons can easily be made as follows:

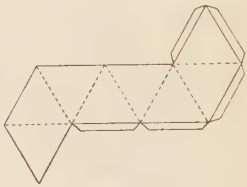
a) draw the diagrams, which are shown below, on stiff paper;
b) cut out these forms following the outside lines;
c) fold along the dashed lines;
d) keep the edges together by pasting down the flaps.



Tetrahedron

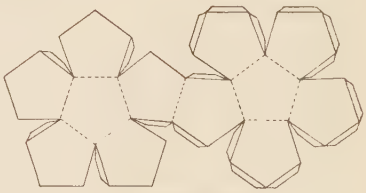


Hexahedron

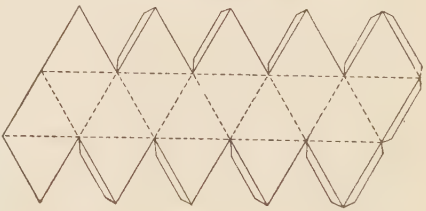


Octahedron

92. Closed Prismatic Surface. The figure generated by a moving straight line that remains parallel to its first position as it continuously intersects a fixed polygon non-coplanar with the line is called a *closed prismatic surface*. The moving line in any of its positions is called an *element* of the surface; an element through a vertex of the fixed polygon is called



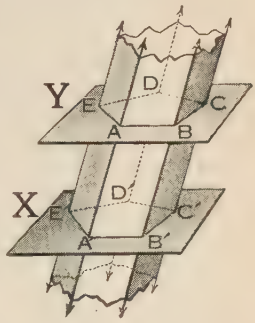
Dodecahedron



Icosahedron

an *edge;* the portion of the surface between any two consecutive edges is called a *face* of the prismatic surface.

93. Prism. The solid formed by the portion of a closed prismatic surface determined by and included between two parallel planes which cut each element of the surface is called a *prism.* The intersections of either plane and the surface is called a *base;* the portions of the edges and faces of the prismatic surface included between the parallel planes are called respectively the *lateral edges* and the *lateral faces* of the prism; the sum of the areas of the lateral faces is called the *lateral area;* the sum of the lateral area and the areas of the two bases is called the *total area;* the perpendicular from any point in the plane of one base to the plane of the other base is called an *altitude* of the prism.



Note: The prismatic surface shown in § 93 is a closed surface. The pupil should understand that not all prismatic surfaces are closed. How would § 92 read, if an open prismatic surface were defined?

94. Right Section. A section of a prism formed by a plane which is perpendicular to a lateral edge and which cuts all the lateral edges, or the lateral edges produced, is called a *right section* of the prism.

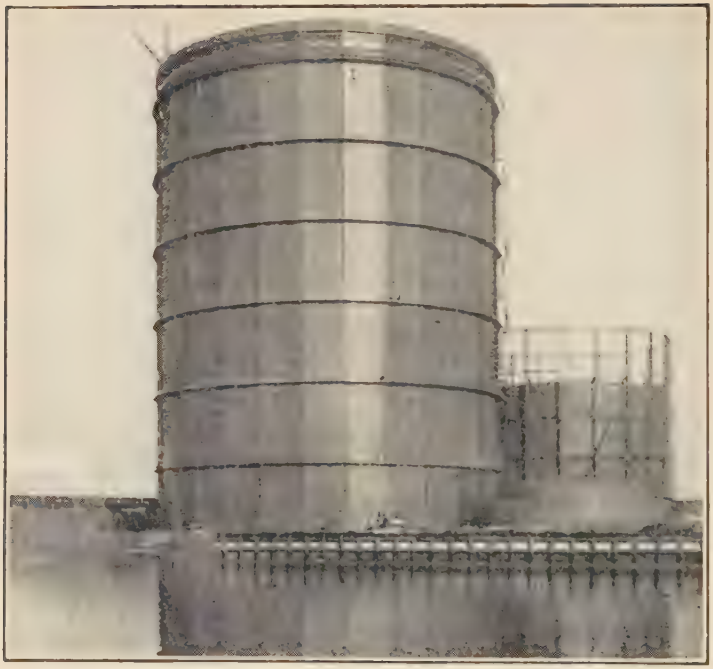
95. Preliminary Theorems. Prisms.

a) *The lateral edges of a prism are equal and parallel.*

b) *The lateral faces of a prism are parallelograms.*

c) *A plane perpendicular to one lateral edge of a prism is perpendicular to all the lateral edges or to the edges produced.*

d) *A section of a prism formed by the plane of any two non-consecutive edges is a parallelogram.*



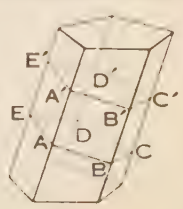
Courtesy Public Service Electric and Gas Co.

A PRISMATIC GAS HOLDER

The picture shows two gas storage holders along the Passaic River at Harrison, N. J. The smaller holder is the usual so-called "gas tank," cylindrical in form and collapsible like a pocket telescope. This holder has a capacity of 3,200,000 cu. ft. The larger gas holder has 28 sides. It is 378 ft. high and 254 ft. 3 in. "across corners" and has a capacity of 15,000,000 cu. ft. This prismatic gas holder is a masterpiece of modern engineering—a triumph for mathematics.

Proposition 1 (C N)

96. Theorem. *Sections of a prism made by parallel planes which cut all the lateral edges are congruent polygons.*



Given: $ABCDE$ and $A'B'C'D'E'$ are sections made by parallel planes cutting all the lateral edges of the prism.

To prove: $ABCDE \cong A'B'C'D'E'$.

Proof: STATEMENTS	FACTS
1. The plane of $ABCDE \parallel$ the plane of $A'B'C'D'E'$.	1. Why?
2. $AB \parallel A'B'$, $BC \parallel B'C'$, etc.	2. § 37.
3. $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc.	3. § 45.
4. $AA' \parallel BB' \parallel CC'$, etc.	4. § 95*a*.
5. $AB = A'B'$, $BC = B'C'$, etc.	5. C 14.
6. $ABCDE \cong A'B'C'D'E'$.	6. Why?

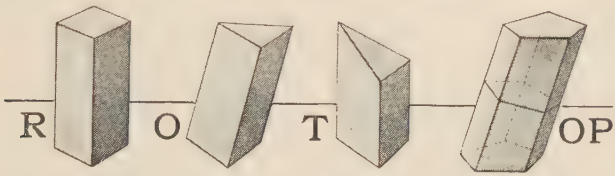
97. Corollary 1. *Right sections of a prism are congruent polygons.*

98. Corollary 2. *The bases of a prism are congruent polygons.*

99. Corollary 3. *Every section of a prism made by a plane parallel to the plane of the base is congruent to the base.*

100. Kinds of Prisms. If a base of a prism is also a right section, the prism is called a *right prism*. If a prism is not a right prism, it is said to be *oblique*. A right prism whose bases are regular polygons is called a *regular prism*. Prisms are also classified according to the form of their bases as *triangular*, *quadrangular*, *pentagonal*, *hexagonal*, *octagonal*, etc.

101. Truncated Prism. The part of a prism included between a base and a section made by a plane oblique to the plane of the base is called a *truncated prism*.



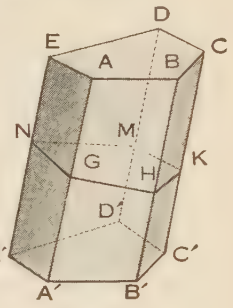
R is a right quadrangular prism; *O* is an oblique triangular prism; *T* is a truncated triangular prism; *OP* is an oblique pentagonal prism with a right section.

102. Preliminary Theorems. Right Prisms.

a) A lateral edge of a right prism equals an altitude.

b) The lateral faces of a right prism are rectangles.

Proposition 2 (C N)



103. Theorem. *The lateral area of a prism equals the product of a lateral edge and the perimeter of a right section.* ($S = ep$.)

Given: A prism with S the lateral area, e a lateral edge, and p the perimeter of a right section $GHKMN$.

To prove: $S = ep$.

Proof:

Statements	Facts
1. e may represent each of the lateral edges (as BB').	1. § 95*a*.
2. Each lateral face (as $ABB'A'$) is a $\square$.	2. § 95*b*.
3. $GH \perp BB'$, $HK \perp CC'$, etc.	3. §§ 94, 95*c*.
4. $ABB'A' = GH \times e$, $BCC'B' = HK \times e$, etc.	4. K 3.
5. $S = e(GH + HK + KM +$ etc.).	5. Ax. of Add.
6. $S = ep$.	6. Sub.

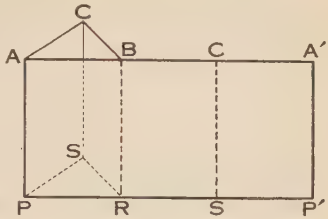
104. Corollary. *The lateral area of a right prism equals the product of its altitude and the perimeter of its base.* ($S = hp$.)

Exercises. Areas of Right Prisms

1. If a rectangular sheet of cardboard $PAA'P'$ is divided into three congruent rectangles, as shown in the annexed figure, on p. 49, and folded along the dashed lines, what figure is formed? In like manner, the lateral surface of a right quadrangular prism may be formed by dividing the rectangular sheet of cardboard into four congruent rectangles. Does the lateral area of each prism thus formed equal the area of the original rectangle? If the rectangular sheets are equal, will

the lateral areas of the two prisms be equal? Is it necessary that the rectangular divisions be congruent for this to be true?

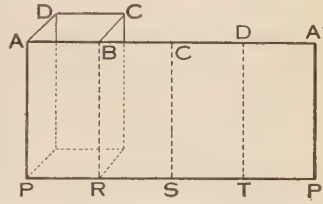
2. If the rectangles $APP'A'$ in Ex. 1 are each 12 in. by 5 in., what is the lateral area of each prism?



3. The total area of each prism in Ex. 1 may be considered as equal to what three figures?

4. If the rectangles $APP'A'$ are each 12 in. by 5 in., what is the total area of each prism?

5. If a right prism with a regular hexagonal base is constructed as in Ex. 1, with $AA' = 12$ in. and $AP = 5$ in., find its total area.



6. Sketch (*a*) a right prism whose base is a scalene triangle; (*b*) a right prism whose base is a right triangle; (*c*) a regular prism with three lateral faces; (*d*) a regular prism with four lateral faces.

7. What measurements would you need to know for each of the prisms in Ex. 6 to find the lateral area? to find the total area?

8. Does the lateral area depend upon the altitude alone? upon a side of the base? Upon what does the lateral area depend?

9. Does the total area of a right prism vary directly as the altitude? as a side of the base? Upon what does the total area depend?

10. Write a formula for the lateral area of a right prism, if the altitude is h and the base is (*a*) an equilateral triangle whose side is s; (*b*) a rectangle whose dimensions are a and b; (*c*) a regular hexagon whose side is s; (*d*) a right triangle whose legs are a and b.

11. Write a formula for the total areas of each of the prisms in Ex. 10.

12. If the altitude and perimeter of the base of a right prism are each doubled, is the lateral area doubled? How is it changed?

13. If the altitude and the perimeter of the base of a right prism are each tripled, how many times is the lateral area increased? How is the total area changed?

In Exs. 14–18, find the lateral area of a right prism whose altitude is 10 and whose base is

14. a triangle whose sides are 5, 12, and 13;

15. an equilateral triangle whose side is 6;

16. a right triangle whose legs are 6 and 8;

17. a 45–45–90 triangle whose legs are each 4;

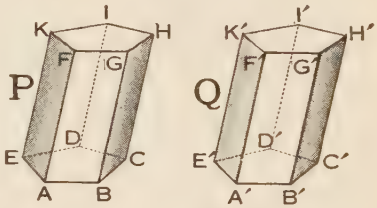
18. a 30–60–90 triangle whose hypotenuse is 8.

19. A right prism has a square base whose side is 6. Find the altitude, (*a*) if the lateral area is 168; (*b*) if the total area is 168.

Proposition 3

105. Theorem. *Prisms with three faces which include a trihedral angle of one congruent to the respective parts of another and similarly placed are congruent.*

Given: Prisms P and Q with AG, AK, and AD congruent to $A'G'$, $A'K'$, and $A'D'$ respectively.



To prove: $P \cong Q$.

Proof:

Statements	Facts
1. $\angle BAF = \angle B'A'F'$, $\angle FAE = \angle F'A'E'$, $\angle BAE = \angle B'A'E'$.	1. Given, G 1.
2. Trihedral $\angle A$ = trihedral $\angle A'$.	2. § 81.
3. Place P upon Q so that trihedral angles A and A' coincide.	3. A 33.
4. AG will coincide with $A'G'$, AK with $A'K'$, and AD with $A'D'$.	4. Def. of congruence.
5. Points K, F, G coincide with points K', F', G' and their planes coincide.	5. § 5*b*.
6. $AD \cong A'D'$, $AD \cong FI$, $A'D' \cong F'I'$.	6. Given, § 98.
7. FI coincides with $F'I'$ and H falls on H', I upon I'.	7. Why?
8. HC coincides with $H'C'$, ID with $I'D'$.	8. Why?
9. $P \cong Q$.	9. Why?

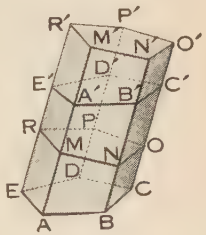
106. Corollary 1. *Right prisms with congruent bases and equal altitudes are congruent.*

107. Corollary 2. *Truncated prisms with three faces which include a trihedral angle of one equal to the respective parts of another and similarly placed are congruent.*

108. Equal Figures. Two figures which are the sums or differences of the same number of congruent figures are said to be *equal*. Such figures are sometimes called equivalent figures.

Proposition 4 (C N)

109. Theorem. *An oblique prism is equal to a right prism whose base is a right section of the oblique prism and whose altitude equals a lateral edge of the oblique prism.*



Given: $E'C$ is an oblique prism whose right section MP is the base of right prism $R'O$, altitude MM' = lateral edge AA'.

To prove: Oblique prism $E'C$ = right prism $R'O$.

Proof: STATEMENTS	FACTS
1. $AA' = MM'$, $A'M = A'M$, $AM = A'M'$ and similarly $BN = B'N'$.	1. Given, Iden., A 7.
2. $AB = A'B'$, $MN = M'N'$.	2. C 3.
3. $\measuredangle$ of face AN = corresponding $\measuredangle$ of face $A'N'$.	3. G 12
4. $AN \cong A'N'$ and similarly $AR \cong A'R'$.	4. Def. of congruence.
5. $AD \cong A'D'$.	5. § 98.
6. Truncated prism $R'C' \cong$ truncated prism RC.	6. § 107.
7. $R'C - R'C' = R'C - RC$ or $E'C = R'O$.	7. A 7, § 108.

110. Parallelopipeds. If a base of a prism is a parallelogram, the figure is called a *parallelopiped;* if a base is a rectangle, the figure is called a *rectangular parallelopiped;* if the parallelopiped is a right prism, the figure is called a *right parallelopiped.* A right rectangular parallelopiped is often called a *rectangular solid.* If three edges of a rectangular solid which meet at a point are equal, the figure is called a *cube.*

111. Preliminary Theorems. Parallelopipeds.

a) *All the faces of a parallelopiped are parallelograms.*

b) (**c N**) *The opposite lateral faces of a parallelopiped are congruent and parallel.*

c) *All the faces of a rectangular solid are rectangles and opposite lateral faces are congruent.*

d) *All the faces of a cube are congruent squares.*

e) *Any pair of opposite faces of a parallelopiped may be taken as bases.*

f) *If a lateral edge of a parallelopiped is perpendicular to a base, the figure is a rectangular solid.*

g) *A right section of a parallelopiped is a parallelogram.*

Exercises. Areas of Parallelopipeds

1. Find the total surface of a crayon box 4 in. by 5 in. by 7 in.

2. Write a formula for the total area (T) of a rectangular solid whose length, width, and height are denoted respectively by l, w, and h.

3. Find the total area of a 6-inch cube.

4. Write a formula for the total area (T) of a cube whose edge is e.

5. The diagonal of a face of a cube is $6\sqrt{2}$. Find the total area of the cube.

6. Find the diagonal of a cube, if each edge is (*a*) 1; (*b*) 2; (*c*) 3; (*d*) e.

7. The diagonal of a cube is $8\sqrt{3}$. Find its total area.

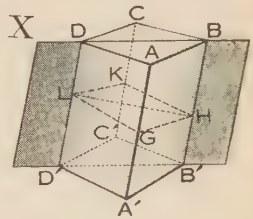
8. If each edge of a rectangular solid is doubled, how many times is the total area increased?

9. Find the edge of a cube whose total area is (*a*) 24; (*b*) 96; (*c*) 216; (*d*) $12\,a^2$.

10. Find the lateral area of a box in the form of a rectangular solid three of whose edges are 2 in., 3 in., and 4 in. respectively. How many answers are possible?

Proposition 5 (C N)

112. Theorem. *A plane through diagonally opposite edges of a parallelopiped divides it into two equal triangular prisms.*



Given: Plane X passes through edges DD' and BB' of parallelopiped DB'.

To prove: Prism A–$A'B'D'$ = prism C–$C'D'B'$.

Proof:

Statements	Facts
1. Let $GHKL$ be $\perp AA'$.	1. § 14.
2. $GHKL$ is a right section.	2. § 94.
3. $GHKL$ is a $\square$.	3. § 111g.
4. $\triangle GHL \cong \triangle KHL$.	4. B 9.
5. $\triangle GHL$ and $\triangle KHL$ are right sections.	5. §§ 95c, 94.
6. Prism A–$A'B'D'$ = a right prism with base GHL and altitude AA'; prism C–$C'D'B'$ = a right prism with base KHL and altitude CC'.	6. § 109.
7. $AA' = CC'$.	7. § 95a.
8. The right prisms are congruent.	8. § 106.
9. Prism A–$A'B'D'$ = prism C–$C'D'B'$.	9. Sub.

113. Unit of Volume. Volume of a Solid. A *unit of volume* is a cube whose edge is a unit of length. The number of units of volume contained in the space occupied by a solid is called the *volume* of the solid.

114. Equal Solids. Two solids which have equal volumes are said to be *equal*.

115. Dimensions of a Rectangular Solid. The lengths of the three edges of a rectangular solid which meet at a vertex

are called its *dimensions*. The pupil should understand that the dimensions, areas, and volumes of figures are numbers.

116. Principle.

The volume of a rectangular solid equals the product of its three dimensions.

If V is the volume of a rectangular solid and if a, b, and c are its dimensions, $V = abc$.

117. Corollary 1. *The volume of a cube equals the cube of its edge.* ($V = e^3$.)

118. Corollary 2. *The volume of a rectangular solid equals the product of the area of its base and its altitude.* ($V = hB$.)

119. Corollary 3. *Rectangular solids have the same ratio as the products of their dimensions.* ($V : V' = abc : a'b'c'$.)

120. Corollary 4. *Rectangular solids with equal altitudes have the same ratio as their bases; rectangular solids with equal bases have the same ratio as their altitudes.*

121. Corollary 5. *Rectangular solids with equal dimensions are equal.*

Exercises. Volumes of Rectangular Solids

1. Find the volume of a cube whose edge is 3; 5; 4 a; $3\sqrt{2}$.

2. Find the volume of a cube whose total area is 150; 300; 30 a^2.

3. What are the dimensions of a cube whose total area and volume are numerically equal?

4. The ratio of the total areas of two cubes is 16 : 9. What is the ratio of their volumes?

5. The volume of a rectangular solid is 1350 and one of its faces is a square whose area is 225. Find the three dimensions.

6. Find the volume of a rectangular solid, if its dimensions are consecutive whole numbers and its total area is 214.

7. Find the approximate number of bricks $9'' \times 3'' \times 4\frac{1}{2}''$ needed for a solid wall 30 ft. long, 8 ft. high, and $14\frac{1}{2}$ in. thick, if 10% of the wall is mortar.

8. How many times is the volume of a rectangular solid increased if (*a*) a is doubled; (*b*) a and b are doubled; (*c*) a, b, and c are doubled; (*d*) the total area is doubled?

9. State three ways of doubling the capacity of a box.

10. Find the weight of a steel plate 6 ft. 6 in. long, 4 ft. wide, and $\frac{3}{4}$ in. thick, if a cubic inch of steel weighs .28 lb.

11. If the dimensions of box are each increased 50%, what is the per cent of increase in volume?

12. How many cubes 1 cm $\times$ 1 cm $\times$ 1 cm are there in a cube whose edge is one inch? (1 in. = 2.54 cm).

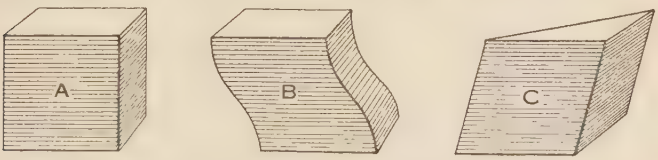
13. What weight of water falls on an acre of ground during a storm in which the rainfall is $\frac{3}{16}$ in? (See the Table of Equivalents, p. 152.)

14. The sum of the squares of the three dimensions of a rectangular solid is 3150 and the dimensions are to each other as 1 : 2 : 3. Find the volume of the solid.

15. The diagonal of a cube is d, what is its volume in terms of d?

122. Cavalieri's Theorem. *If two solids have equal altitudes and if the sections formed by planes parallel to and at equal distances from their respective bases are equal, the solids are equal.*

The proof of this theorem can be made rigorous only by principles of advanced mathematics. The reasonableness of the theorem may be appreciated from the following discussion. Suppose the figures given here represent three piles of cards of uniform thickness, that each pile contains



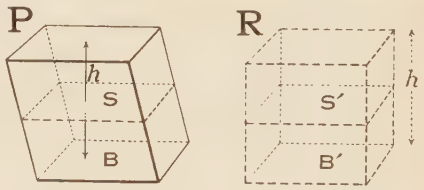
n cards and that the surface area of each card is a square inches. Although the cards may have different shapes and the piles may vary in form, the piles will have equal volumes. Moreover, the cards need not all be equal in area for the

piles to have equal volumes. In any case, if the cards at the same distance from their respective bases are equal, the volumes of the piles are equal.

This theorem was used for the first time by Bonaventura Cavalieri (1598–1647), a professor of mathematics at the University of Bologna, Italy.

Proposition 6 (c N)

123. Theorem. *The volume of a parallelopiped equals the product of a base and its altitude.*



Given: P is any parallelopiped with base B, altitude h, and volume V.

To prove: $V = Bh$.

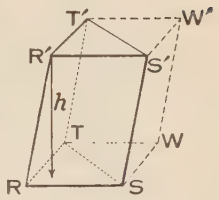
Proof:

Statements	Facts
1. Let rectangle $B' = \square B$ and at the vertices of B' erect $\perp$s to B' each equal to h; join the ends of these $\perp$s to form a rectangular solid R with volume V'.	1. § 18, A 24, A 27, § 110.
2. Let sections S and S' be formed by planes parallel to and at equal distances from the respective bases of P and R.	2. §§ 41, 8.
3. $S = B$, $S' = B'$, $S = S'$.	3. § 99, Sub.
4. $P = R$.	4. § 122.
5. $V' = B'h$, $V = Bh$.	5. § 118, Sub.

Note: The pupil should observe that in the formula for the preceding theorem and in formulas which follow, lines are denoted by small letters while areas and volumes are denoted by capital letters. The statement that B is the base means in full that B is the area of the base.

Proposition 7 (C)

124. Theorem. *The volume of a triangular prism equals the product of a base by its altitude.*



Given: R'–RST is a triangular prism with base B (or RST), altitude h, and volume V.

To prove: $V = Bh$.

Proof:

Statements	Facts
1. Let planes through SS' and TT' parallel respectively to $R'T$ and $R'S$ intersect in WW', and let them also intersect the planes of RST and $R'S'T'$ in SW, WT, $S'W'$, and $W'T'$ respectively forming parallelopiped $R'W$.	1. §§ 41, 8, 37, 110.
2. $\square\ RSWT = 2 \times \triangle\ RST = 2\,B$.	2. Why?
3. Volume of $R'W = 2\,Bh$.	3. Why?
4. Volume of R'–$RST = \frac{1}{2}$ volume of $R'W$.	4. § 112.
5. $V = \frac{1}{2} \times 2\,Bh = Bh$.	5. Why?

125. Corollary 1 (c N). *The volume of any prism equals the product of a base by its altitude.*

Hint: Divide the given prism into triangular prisms.

126. Corollary 2. *If V and V' denote the volumes of two prisms with bases B and B' and altitudes h and h':*

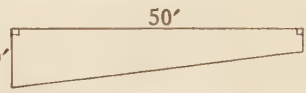
a) $V : V' = Bh : B'h'$;
b) $V : V' = B : B'$, *when* $h = h'$;
c) $V : V' = h : h'$, *when* $B = B'$;
d) $V = V'$, *when* $B = B'$ *and* $h = h'$.

Note: The pupil should state in full each part of Corollary 2 and indicate the proofs.

Exercises. Volumes of Right Prisms

In Exs. 1–9 find the volume of a right prism whose altitude is 10 and whose base is

1. a square whose diagonal is $7\sqrt{2}$;
2. an equilateral triangle whose side is 4;
3. a triangle whose sides are 5, 12, and 13;
4. a right triangle whose legs are 6 and 8;
5. an isosceles triangle whose sides are 5, 5, and 6;
6. a 45–45–90 triangle whose legs are 4;
7. a 30–60–90 triangle whose shorter leg is 6;
8. a rhombus whose diagonals are 6 and 10;
9. a regular hexagon whose side is 6.
10. Find the volume of a swimming tank whose width is 30 ft. and whose sides are trapezoids as shown in the annexed figure.



HINT: Consider the tank as a right prism with a trapezoidal base.

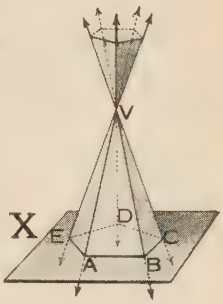
11. Find the volume of the V-shaped watering trough shown in the annexed figure, if the top is 2 ft. by 6 ft. and the depth is 1 foot.

12. How many gallons of water are in the trough in Ex. 11, if the water is 9″ deep?
13. Given V the volume and h the altitude of a right prism whose base is a regular hexagon of side s. Find s in terms of V and h.

127. Closed Pyramidal Surface. The figure generated by a moving straight line that continuously intersects a fixed polygon as the line passes through a fixed point non-coplanar with the polygon is called a closed *pyramidal surface.* The fixed point is called the *vertex;* the moving line in any of its positions is called an *element;* an element through any vertex of the polygon is called an *edge;* the portion of the surface between any two consecutive edges is called a *face;* the two parts of the surface which have only the vertex in common are called *nappes.*

128. Pyramid. The solid formed by the portion of a nappe of a closed pyramidal surface determined by and in-

cluded between the vertex and a plane which cuts each element of the surface is called a *pyramid*. The vertex of the surface is also the vertex of the pyramid; the intersections of the plane and the surface is called the *base;* the portions of the edges and the faces of the pyramidal surface included between the vertex and the intersecting plane are called respectively the *lateral edges* and the *lateral faces* of the pyramid; the perpendicular from the vertex to the plane of the base is called the *altitude; lateral area* and *total area* as defined in § 93 apply to pyramids and to all other polyhedrons.



129. Kinds of Pyramids. Pyramids are classified according to the form of their bases, as *triangular, quadrangular, square, hexagonal,* etc. A pyramid whose base is a regular polygon and whose altitude meets the base at its center is called a *regular pyramid.*

130. Preliminary Theorems. Regular Pyramids.

a) The lateral edges of a regular pyramid are equal.

b) The lateral faces of a regular pyramid are congruent isosceles triangles.

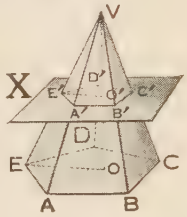
c) The face angles at the vertex of a regular pyramid are equal.

d) The altitudes of the lateral faces of a regular pyramid are equal, if they include the vertex of the pyramid.

131. Slant Height of a Regular Pyramid. The altitude of any one of the lateral faces of a regular pyramid which includes the vertex of the pyramid is called the *slant height.*

Proposition 8 (C N)

132. Theorem. *If a pyramid is cut by a plane parallel to its base, the lateral edges and the altitude are divided proportionally and the section formed is similar to the base.*



Given: Pyramid $V\text{–}ABCDE$ is cut by plane X parallel to its base, forming section $A'B'C'D'E'$ and cutting the altitude VO in O'.

To prove: $\frac{VO'}{O'O} = \frac{VA'}{A'A} = \frac{VB'}{B'B}$ = etc. and $A'B'C'D'E' \sim ABCDE$.

Proof:

Statements	Facts
1. $X \parallel ABCDE$.	1. Why?
2. $E'O' \parallel EO, A'B' \parallel AB, B'C' \parallel BC$, etc.	2. § 37.
3. $\frac{VO'}{O'O} = \frac{VA'}{A'A} = \frac{VB'}{B'B}$ = etc.	3. M 6, Sub.
4. $\triangle A'VB' \sim \triangle AVB$, $\triangle B'VC' \sim \triangle BVC$, etc.	4. M 7, N 5.
5. $\frac{A'B'}{AB} = \left(\frac{VB'}{VB}\right) = \frac{B'C'}{BC} = \left(\frac{VC'}{VC}\right)$, etc.	5. M 10.
6. $\angle E'A'B' = \angle EAB$, $\angle A'B'C' = \angle ABC$, etc.	6. § 45.
7. $A'B'C'D'E' \sim ABCDE$.	7. Why?

133. Corollary 1 (N). *The base of a pyramid and a section parallel to the base have the same ratio as the square of the altitude of the pyramid and the square of the perpendicular from the vertex to the section.*

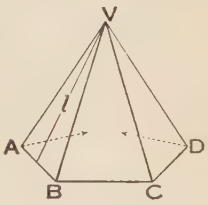
Hint: $A'B'C'D'E' : ABCDE = (E'A')^2 : (EA)^2$ [M 13]; $E'A' : AE = VE' : VE = VO' : VO$ [M 10].

134. Corollary 2. *If two pyramids have equal altitudes and equal bases, sections parallel to the bases and equidistant from the vertices are equal.*

HINT: Let the bases be denoted by B and B', the sections by S and S', the altitude by h, and the distances from the vertices to the sections by h'. $\frac{S}{B} = \frac{h'^2}{h^2} = \frac{S'}{B'}$ (§ 133 and Sub.); $B = B'$ (Given); $S = S'$ (A 9).

Proposition 9 (C N)

135. Theorem. *The lateral area of a regular pyramid equals one-half the product of its slant height and the perimeter of its base.*



Given: V–$ABCD\ldots.A$ is a regular pyramid of n lateral faces with lateral area S, slant height l, and perimeter of the base p.

To prove: $S = \frac{1}{2}lp$.

HINT: $\triangle VAB = \frac{1}{2}l \times AB$, $\triangle VBC = \frac{1}{2}l \times BC$, etc. Why?
$\triangle VAB + \triangle VBC + \text{etc.} = \frac{1}{2}l(AB + BC + \text{etc.})$. Why?
$AB + BC + \text{etc.} = p$. Why? $S = \frac{1}{2}lp$. Why?

Exercises. Areas of Regular Pyramids

1. The lateral faces of a regular pyramid as shown in Figure 1 could be arranged in the triangular sections as in Figure 2. This series of triangles, in turn, is equal to a single triangle (Fig. 3) whose altitude is the slant height and the length of whose base equals the perimeter of the regular pyramid. May the lateral area of any pyramid be found in this manner? The total area of a regular pyramid is equal to the area of what two figures when this method is used?

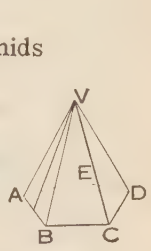


Fig. 1

2. If the slant height is denoted by l and a side of the base by s, write a formula for the total area of a regular pyramid whose base is (*a*) an equilateral triangle; (*b*) a square; (*c*) a regular hexagon.

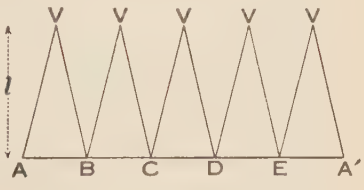


Fig. 2

3. If one edge of a regular tetrahedron is denoted by e, find the total area.

4. State three ways by which the lateral area of a regular pyramid may be quadrupled.

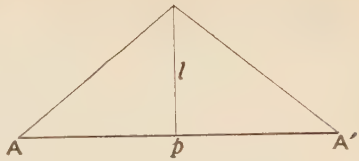


Fig. 3

In Exs. 5–7, find the lateral area of a regular pyramid whose slant height is 6 and whose base is

5. a square whose side is 4;

6. an equilateral triangle whose side is 4;

7. a regular hexagon whose side is 4.

136. Truncated Pyramid—Frustum of a Pyramid. The part of a pyramid included between the base and a section made by a plane cutting all the lateral edges is called a *truncated pyramid;* if the section is parallel to the base, the figure is called a *frustum of the pyramid.* The base and the section are called respectively the *lower* and *upper bases* of the frustum; a perpendicular from any point in one base to the plane of the other base is called an *altitude.*

NOTE: The pupil should note that the theorems and formulas of frustums of pyramids are for frustums of regular pyramids only.

137. Preliminary Theorems. Frustum of a Regular Pyramid.

a) The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.

b) The altitudes of the lateral faces of a frustum of a regular pyramid are equal.

138. Slant Height of a Frustum of a Regular Pyramid. The altitude of any one of the lateral faces of a frustum of a regular pyramid is called the *slant height* of the frustum.

139. Corollary (c n). *The lateral area of a frustum of a regular pyramid equals one-half the product of its slant height and the sum of the perimeters of its bases.* $S = \frac{1}{2} l(p + p')$.

Exercises. Areas of Frustums of Pyramids

1. What formula of plane geometry corresponds with the formula for the lateral area of a frustum of a regular pyramid?

2. Find the lateral area of a frustum of a regular pyramid, if l, p, and p' are respectively, (*a*) 2, 6, and 8; (*b*) 4, 6, and 10; (*c*) 6, 5, and 9; (*d*) 5, 6, and 11.

3. Find the lateral area of a frustum of a regular pyramid with 8 as its slant height and with square bases whose sides are respectively 5 and 7. Find the total area.

4. Find the lateral area of a frustum of a regular pyramid, if 10 is its slant height and if its bases are equilateral triangles with 4 and 6 as their sides. Find the total area of the frustum.

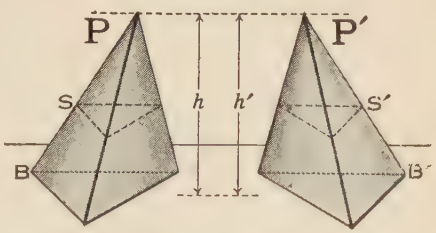
5. Each of the lateral faces of a frustum of a regular pyramid with square bases is an isosceles trapezoid whose sides are 4, 5, 10, and 5. Find the total area of the frustum.

6. Given a frustum of a regular square pyramid the sides of whose bases are s and s' and whose slant height is l, prove that the total area equals $s^2 + s'^2 + 2\,l(s + s')$.

Proposition 10 (c N)

140. Theorem. *Triangular pyramids with equal altitudes and equal bases are equal.*

Given: $\boldsymbol{P}$ and $\boldsymbol{P'}$ are triangular pyramids with altitude h = altitude h' and base B = base B'.



To prove: $\boldsymbol{P} = \boldsymbol{P'}$.

Proof:

Statements	Facts
1. Let S and S' be any two sections of $\boldsymbol{P}$ and $\boldsymbol{P'}$ which are parallel to their respective bases and at equal distances from them.	1. § 41.
2. $h = h'$, $B = B'$.	2. Given.
3. $S = S'$.	3. § 134.
4. $\boldsymbol{P} = \boldsymbol{P'}$.	4. § 122.

Exercises. Pyramids

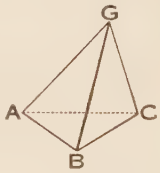
1. Find the lateral area and total area of a regular square pyramid whose lateral edges are 5, and each side of whose base is 6.

2. The sides of the bases of the frustrum of a regular pyramid are 6 and 18 and the slant height is 8. What is the slant height of the entire pyramid? What is the ratio of the altitudes of the two figures?

3. Find the lateral area of a regular square pyramid, assuming suitable values for (*a*) the altitude and the slant height; (*b*) the slant height and a lateral edge; (*c*) the altitude and a side of the base; (*d*) the slant height and a side of the base; (*e*) the altitude and a lateral edge.

Oral Exercises

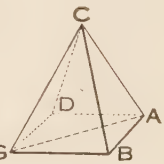
1. How many faces has a triangular pyramid (such as *G–ABC*)? How many points may be taken as a vertex of the pyramid? How many faces may be taken as a base?



2. The triangular pyramid in Ex. 1 was named *G–ABC*. Name it in three other ways.

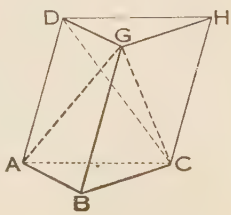
3. How many faces has a quadrangular pyramid (such as *C–ABGD*)? How many faces may be taken as its base?

4. Suppose the pyramid in Ex. 3 is divided into two triangular pyramids by a plane *CGA*. In how many ways can each of the triangular pyramids be named? Name two triangular pyramids which have a common vertex and whose bases are equal triangles in the same plane. Why are two such triangular pyramids equal?



5. In the triangular prism *DGH–ABC*, planes *CGA* and *DGC* form three triangular pyramids. Read pyramid *C–GAD* using *G* as its vertex.

6. Are pyramids *G–DCA* and *C–GAD* identical? Does *G–DCA* = *G–DCH*? Does *C–GAD* = *C–GAB*?

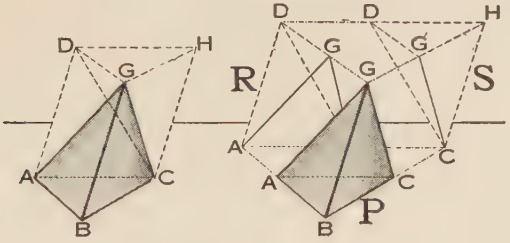


Proposition 11 (C)

141. Theorem. *The volume of a triangular pyramid equals one-third the product of its base and its altitude.*

Given: G–ABC is a triangular pyramid with volume V, altitude h, and base B.

To prove: $V = \frac{1}{3}Bh$.



Proof:

Statements	Facts
1. Let AD and CH be parallel to BG.	1. Why?
2. Pass planes through BG and CH, BG and AD, AD and CH.	2. Why?
3. Through G pass a plane $\parallel ABC$ forming triangular prism GDH–ABC and draw DC.	3. Why?
4. Planes GDC and GAC divide the prism into three triangular pyramids. ($\boldsymbol{P}$, $\boldsymbol{R}$, and $\boldsymbol{S}$.)	4. Def.
5. $\triangle AGB \cong \triangle AGD$, $\triangle CDA \cong \triangle CDH$.	5. B 9.
6. C–AGB = C–AGD (or $\boldsymbol{P} = \boldsymbol{R}$) and G–CDA = G–CDH (or $\boldsymbol{R} = \boldsymbol{S}$).	6. § 140.
7. $\boldsymbol{P} = \boldsymbol{R} = \boldsymbol{S} = \frac{1}{3}$ prism GDH–ABC.	7. Sub.
8. GDH–ABC = Base × altitude.	8. § 124.
9. $V = \frac{1}{3}Bh$.	9. Sub.

Note. Pupils should demonstrate this theorem using only the single figure; the separated prism was given to aid pupils in visualizing the three triangular pyramids. It should be noted that $\boldsymbol{R}$ is considered first as C–AGD and then as G–CDA and that the equality of all three pyramids is proved by showing that $\boldsymbol{P}$ and $\boldsymbol{S}$ are each equal to $\boldsymbol{R}$.

142. Corollary 1 (c N). *The volume of any pyramid equals one-third the product of its base and its altitude.*

HINT: Divide the pyramid into triangular pyramids.

143. Corollary 2. *If V and V' denote the volumes of two pyramids with their respective bases B and B' and altitudes h and h',*

a) $V : V' = Bh : B'h'$;

b) $V : V' = B : B'$, *when* $h = h'$;

c) $V : V' = h : h'$, *when* $B = B'$;

d) $V = V'$, *when* $B = B'$ *and* $h = h'$.

NOTE: The pupil should state in full each part of Corollary 2 and indicate the proofs.

Exercises. Volumes of Pyramids

Find the volume of a pyramid whose altitude is 10 and whose base is

1. a square whose sides are 6;

2. an equilateral triangle whose sides are 6;

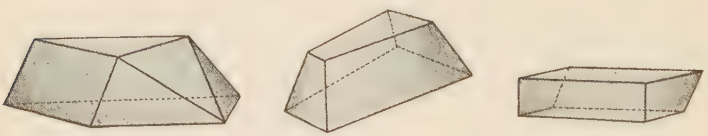
3. a right triangle whose legs are 5 and 12;

4. a rhombus whose diagonals are 6 and 8;

5. a regular hexagon whose sides are 6.

6. What is the volume of a regular square pyramid, if each side of the base is 24 and (*a*) if the altitude is 18; (*b*) if the slant height is 15; (*c*) if a lateral edge is 20.

144. Prismatoid. A polyhedron two of whose bounding faces are polygons in parallel planes and whose other faces are triangles, trapezoids, parallelograms, or combinations



Prismatoids

of them is called a *prismoid* or a *prismatoid*. The word *prismatoid* means "like a prism." The faces in parallel planes are called *bases;* the other faces are called *lateral faces;*

the perpendicular from any point in the plane of one base to the plane of the other base is called an *altitude;* the section which is equidistant from the two bases and parallel to one of them is called the *mid-section.*

145. Volume of a Prismatoid (n). If V denotes the volume, h the altitude, B and B' the areas of the bases, and M the area of the mid-section of a prismatoid,

$$V = \tfrac{1}{6}\, h(B + B' + 4\, M).$$

NOTE: This formula will be assumed without proof.

146. Similar Polyhedrons. Polyhedrons of the same number of faces whose like-placed faces are similar polygons and whose corresponding polyhedral angles are equal are called *similar polyhedrons.*

Exercises. Prismatoids

1. If b is a side of a triangle and m is a line which bisects the other two sides, find m if b equals 6; 10; 18; x.

2. In a regular quadrangular pyramid, find a side (m) of the mid-section (M), if each side of the base is 8; 12; 20; b.

3. Find the area of the mid-section for each part of Ex. 2.

4. In any pyramid, what is the ratio of the area of the mid-section to the area of the base? Find the area of the mid-section of a pyramid, if its base is 12 sq. in.; 20 sq. in.; B sq. in.

5. Find the median of a trapezoid whose bases are 5 and 7; 6 and 10; 8 and 18; a and b.

6. Find the area of the mid-section of a frustum of a regular square pyramid, if the sides of its bases are 5 and 7; 6 and 10; 8 and 18; a and b.

7. Show that when a prismatoid is a right prism the prismatoid formula reduces to $V = Bh$.

8. Prove the formula for the volume of a pyramid by assuming that the upper base of a prismatoid equals zero.

9. Show that when a prismatoid is a frustum of a regular square pyramid, the prismatoid formula reduces to $V = \tfrac{1}{3}\, h(B + B' + \sqrt{B}\overline{B'})$.

HINT: Let $\sqrt{B}$ and $\sqrt{B'}$ be the sides of the respective bases.

10. Use the prismatoid formula to find the volume of a frustum of a square pyramid whose height is 6 and the sides of whose bases are (*a*) 10 and 14; (*b*) 10 and 18; (*c*) 12 and 20; (*d*) 5 and 9.

11. By the prismatoid formula, find the volume of a frustum of a pyramid whose altitude is 12 and whose bases are rectangles 20×16 and 10×8.

12. A feed box has a rectangular base $4' \times 1'$ and a rectangular cover $5' \times 2'$, inside measurements. Find the volume when the box is one foot deep.

147. Miscellaneous Theorems. The following theorems may be assumed without proof and applied in the exercises which follow:

a) **(n)** *The volume of a frustum of a pyramid equals one-third the altitude multiplied by the sum of the bases and their mean proportional.*

$V = \frac{1}{3} h(B + B' + \sqrt{BB'})$ (See p. 134 for proof.)

b) **(n)** *The volumes of two tetrahedrons with a trihedral angle of the one equal to a trihedral angle of the other have the same ratio as the products of the edges of the equal trihedral angles.* (See p. 135 for proof.)

c) **(n)** *The volumes of similar tetrahedrons have the same ratio as the cubes of their respective edges.* (See p. 136 for proof.)

d) **(n)** *Similar polyhedrons can be separated into like-placed similar tetrahedrons.*

e) **(c)** *In similar polyhedrons, the ratio of the total areas equals the ratio of the squares of corresponding edges;* **(c n)** *and the ratio of their volumes equals the ratio of the cubes of corresponding edges.*

f) **(c)** *Prisms with three faces including a trihedral angle of one congruent to the respective parts of another and similarly placed are congruent.*

g) **(n)** *In a convex polyhedron, if F denotes the number of faces, E the edges, and V the vertices,*

$$E + 2 = V + F.$$

NOTE: This formula was possibly known to the early Greek geometers. Later it was stated and proved by Descartes and Euler. The formula finds use in the study of crystals.

Exercises. Volumes of Frustums of Pyramids

1. Find the volume of a frustum of a pyramid, if the altitude and bases are (*a*) 6, 9, and 25; (*b*) 9, 25, and 36; (*c*) 12, 49, and 64; (*d*) 16, 81, and 100.

2. If the volume of the frustum of a square pyramid is 152 cu. in. and the sides of the bases are 4 and 6 in., respectively, find the altitude of the frustum.

3. A regular pyramid has an altitude of 24 in. and a square base with 16 in. on a side. If a plane parallel to the base bisects the altitude, find the volume of the frustum thus formed.

4. If the pyramid in Ex. 3 is divided into four parts by three planes which are parallel to the base and which divide the altitude into four equal parts, find the volume of each part of the pyramid.

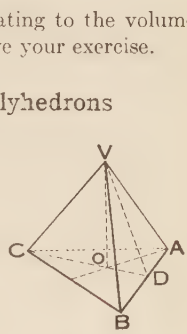
5. Check the results of Ex. 4 by comparing the sum of the parts with the volume of the pyramid.

6. The bases of a frustum of a regular pyramid are equilateral triangles whose sides are 6 and 4. Find the volume of the frustum, if the altitude is 12.

7. State a problem of your own composition relating to the volume of a frustum of a regular hexagonal pyramid. Solve your exercise.

Exercises. Volumes of Regular Polyhedrons

1. Find the volume of a regular tetrahedron whose edges are 12, by finding in order VD, OD, VO, the area of $\triangle ABC$, and the volume of $V\text{–}ABC$.



2. If e is the edge of a regular tetrahedron, show that its slant height is $\frac{e}{2}\sqrt{3}$, its altitude is $\frac{e}{3}\sqrt{6}$, and its volume is $\frac{e^3}{12}\sqrt{2}$.

3. Check the results of Ex. 1 by the formulas of Ex. 2.

4. If the area if a regular tetrahedron is $36\sqrt{3}$, find its volume.

5. The volume of a regular tetrahedron is $18\sqrt{2}$. Find its total area.

6. Find the total area and the volume of a regular hexahedron whose edge is 12.

7. Find the volume of a regular hexahedron whose diagonal is $a\sqrt{3}$.

8. Find the edge of a regular hexahedron, if the numerical ratio of its volume and its total area is 5 : 2.

9. Find the diagonal of a regular hexahedron, if its volume and total area are numerically equal.

10. Find the volume of a regular octahedron whose edge is 6.

11. If e is the edge of a regular octahedron, show that its total area is $2e^2\sqrt{3}$ and that its volume is $\frac{e^3}{3}\sqrt{2}$.

12. Find the edge of a regular octahedron whose volume is $576\sqrt{2}$.

Exercises. Similar Polyhedrons

1. Prove that two polyhedrons which are similar to a third polyhedron are similar to each other.

2. A rectangular solid is $6 \times 8 \times 10$. If the shortest edge of a similar solid is 9, what are its other dimensions? What is the ratio of their total areas? What is the ratio of their volumes?

3. Two corresponding dimensions of two similar tetrahedrons are 2 ft. and 3 ft. If the volume of the smaller is 216 cu. ft., find the volume of the larger. If the volume of the larger is 216 cu. ft., find the volume of the smaller.

4. The diagonals of two cubes are a and $a\sqrt{3}$. What is the ratio of their areas? of their volumes?

5. If the edge of a cube is e, what is the edge of a cube twice as large? What is the edge if the surface is twice as large?

6. The volumes of two similar pyramids are 16 and 128. If the altitude of the smaller is 6, what is the altitude of the larger? If the area of the base of the smaller is 4 square units, what is the area of the base of the larger?

Exercises. Equal Polyhedrons

1. A cube whose edge is 12″ equals a right prism whose rectangular base is $16'' \times 18''$. Find the height of the prism.

2. Find the difference between the total areas of the figures in Ex. 1.

3. A prism and a pyramid have equal volumes and equal bases. What is the ratio of their altitudes?

4. A pyramid with a square base 12×12 has an altitude of 8. What is the altitude of a triangular pyramid of equal volume, if the base is an equilateral triangle whose side is 20?

5. The bases of two equal prisms have the ratio of 1 : 9. If the altitude of the first is 12, find the altitude of the second. If the altitude of the second is 12, find the altitude of the first.

6. A cubic foot of iron is rolled into a plate 18 in. wide and $\frac{1}{2}$ in. thick. How long is the plate?

7. Draw a sketch which will illustrate, if 3 ft. = 1 yd., that 27 cu. ft. = 1 cu. yd.

8. Draw a sketch which will illustrate that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

9. Compose a problem of your own relating to a regular pyramid which is cut into two equal parts by a plane parallel to the base.

Miscellaneous Propositions. Polyhedrons

1. Sections of a prism formed by planes parallel to a base are congruent to the base.

2. Sections of a prism formed by parallel planes cutting all the lateral edges are congruent polygons.

3. If a plane divides the lateral edges of a pyramid proportionally, the plane is parallel to the base.

4. A truncated triangular prism is equal to the sum of three pyramids having as a common base the lower base of the prism and having as their vertices the vertices of the upper base of the truncated prism.

5. The volume of a truncated right triangular prism is equal to one-third the product of the base and the sum of the lateral edges.

6. Two tetrahedrons are similar when three faces including a trihedral angle of one are similar to the respective parts of the other.

7. The volume of any truncated triangular prism equals one-third the area of a sight section multiplied by the sum of the lateral edges.

8. In similar polyhedrons, the ratio of the areas of faces equals the ratio of the squares of corresponding edges.

9. If a pyramid is cut by a plane parallel to the base, the pyramid cut off is similar to the original pyramid.

10. If the sum of all the angles of all the faces of any convex polyhedron is denoted by S and the number of vertices by V $S = (V - 2)$ st. $\measuredangle$.

Review Oral Exercises

1. Name four kinds of polyhedrons classified according to the number of faces.

2. Name four kinds of prisms.

3. Name two kinds of parallelopipeds. Is a cube a parallelopiped? Is it a hexahedron?

4. Is every polyhedron a prism? Is every prism a polyhedron?

5. How many faces of a rectangular solid are rectangles?

6. How many faces has a prism whose bases are hexagons? octagons? decagons?

7. What is the edge of a cube whose volume is numerically equal to n times the total area?

8. If the altitude of a rectangular solid is doubled, is the total area doubled? is the volume? How many times is the volume increased, if each dimension is doubled?

9. How many regular polyhedrons are there? Explain the difference between a regular polyhedron, a regular prism, and a regular pyramid.

10. What is the number of diagonals of a parallelopiped? What is the number of diagonals in all the faces of a parallelopiped?

11. May a solid be bounded by four pairs of parallel planes? by five pairs? by six pairs? Name the possible solids.

12. To find the total area of a regular pyramid, how many measurements would you make and what are they, if the base were triangular? quadrangular? Could the volume be calculated from these measurements?

13. What formula of Chapter Two is in the form of $X = \frac{1}{2}\, yz$? What areas of plane geometry may be found by such a formula?

14. What formula of Chapter Two is in the form of $X = \frac{1}{2}\, y(z + w)$? What area of plane geometry may be found by such a formula?

15. The lateral faces of what polyhedrons are equal and parallel?

16. What is a right section? State a theorem which uses the words *right section*. This theorem is used to prove what other theorem?

17. State all the Facts you can about the bases of a prism.

18. State the formula for the lateral area of a prism. How do you find its total area?

19. Under what conditions are prisms congruent?

20. State all the Facts you can about the opposite faces of a parallelopiped.

21. How would you divide a prism into triangular prisms? Are they all equal?

22. State the formula for the volume of a prism.

23. State all the Facts you can about the lateral faces of a regular pyramid.

24. Where does the altitude of a regular pyramid meet the base of the pyramid? What kind of triangle is formed by the altitude, a slant height, and the line joining their feet? by the altitude, a lateral edge, and the line joining their ends? by a slant height, a lateral edge of the same face, and the line joining their ends?

25. State the formulas for the lateral area and volume of a regular pyramid.

26. State the formulas for the lateral area and volume of a frustum of a regular pyramid.

27. State the Prismatoid Formula.

28. In a regular quadrangular pyramid, let a side of the base be denoted by a, the altitude by h, the slant height by l, and each lateral edge by e. Which of these values must you know, if you are to find the volume? In how many ways can the question be answered?

29. If a block of wood, $2'' \times 3'' \times 4''$, were cut down to a pyramid with the same base and altitude, how many cubic inches of wood are cut away? Is there more than one answer? Why?

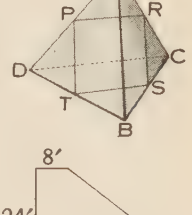
30. Is it possible that the lateral area of a pyramid may be fifty or more times that of another, if the pyramids are equal in volume?

31. What part of the volume of a cube is a pyramid whose base is the base of the cube and whose vertex is at the center of the cube?

Miscellaneous Exercises

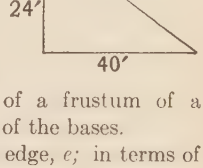
1. Verify the formula, $F + V = E + 2$ for a tetrahedron and a cube.

2. In the figure, A–BCD is a tetrahedron and P, R, S and T are mid-points of four of its edges as shown. Prove that $PRST$ is a parallelogram.



3. During a rainfall of $\frac{3}{16}''$, how many gallons of water could be collected from a flat roof whose area is 3000 sq. ft.?

4. The figure shown represents a cross section of a retaining wall of a reservoir. Find the number of cubic yards of earth in the wall, if it is 270 ft. long.



5. Prove: The perimeter of the mid-section of a pyramid equals one-half the perimeter of the base.

6. Prove: The perimeter of the mid-section of a frustum of a pyramid equals one half the sum of the perimeters of the bases.

7. Express the volume of a cube in terms of its edge, e; in terms of a diagonal of a face, d; in terms of the area of a face, A; in terms of the total area, T.

8. If the areas of three faces of a rectangular solid are 56, 84, and 96, what is its volume?

9. Find the lateral area of a regular quadrangular pyramid, if each side of the base is 10 and the slant height forms an angle with the altitude of (*a*) 30°; (*b*) 45°; (*c*) 60°; (*d*) 70°.

10. The lengths of two boxes have the ratio 1 : 2, the widths have the ratio 2 : 3, and the heights have the ratio 3 : 4. Find the ratios of their total areas and volumes, if the ratios of the dimensions are taken in the same order.

11. The volume of a pyramid, whose lateral edges are three edges of a cube which are concurrent, is what part of the volume of the cube? Prove your answer.

12. The altitude and each side of the base of a regular square pyramid are a. Find the total area and volume in terms of a.

13. The diagonals of a cube divide it into how many pyramids? If the edge of the cube is denoted by e, find the total area and volume of each pyramid.

14. If the centers of the six faces of a cube are joined by line segments none of which are parallel to the edges of the cube, what kind of a polyhedron is formed? If the edge of the cube is denoted by e, find the total area and volume of the polyhedron.

15. How many cubic yards of earth were removed in digging the excavation for an artificial pond 6 ft. deep, if the bottom of the pond is a rectangle 36 ft. $\times$ 48 ft. and the top is also a rectangle 42 ft. $\times$ 56 ft.?

16. If the edges of a regular tetrahedron, a regular octahedron and a regular hexahedron are each denoted by e, show that the ratio of their total areas is $1 : 2 : 2\sqrt{3}$.

17. Show that the volumes of the regular polyhedrons of Ex. 16 have the ratio $1 : 4 : 6\sqrt{2}$.

18. If planes parallel to the base of a pyramid pass through the trisection points of the altitude, find the ratio of the lateral areas and the ratio of the volumes of the pyramids thus formed.

19. The Washington Monument, the highest piece of masonry in the world, is a shaft 500 ft. high (excluding the pyramidal top), 55 ft. square at the base, and 34 ft. square at the top. Find the lateral area.

20. The walls of the Washington Monument are 15 ft. thick at the base and taper to 18 in. thick at the top. Find the number of cubic yards of masonry in the shaft.

CHAPTER THREE

CYLINDERS AND CONES

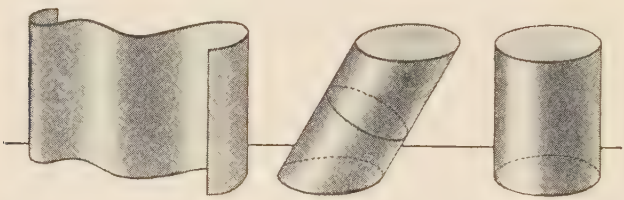
148. Closed Cylindrical Surface. The figure generated by a moving straight line that remains parallel to its first position as it continuously intersects a fixed closed curve non-coplanar with the line is called a *closed cylindrical surface*. The moving line in any of its positions is called an *element* of the surface.

149. Cylinder. The solid formed by the portion of a closed cylindrical surface determined by and included between two parallel planes which cut each element of the surface is called a *cylinder*. The intersection of either plane and the surface is called a *base;* the portion of an element of the cylindrical surface between the bases is called an *element* of the cylinder; the portion of the cylindrical surface between the bases is called the *lateral* or *curved surface;* the sum of the area of the lateral surface and the areas of the bases is called the *total area;* the perpendicular from any point in the plane of one base to the plane of the other base is called an *altitude* of the cylinder.

150. Right Section. A section of a cylinder formed by a plane which is perpendicular to an element of the cylinder and which cuts all the elements, or the elements produced, is called a *right section* of a cylinder.

151. Kinds of Cylinders. If a base of a cylinder is also a right section, the cylinder is called a *right cylinder*, otherwise it is said to be *oblique*. A cylinder whose bases are circles is called a *circular cylinder*. A right circular cylinder is

also called a *cylinder of revolution*, since it may be generated by revolving a rectangle about one of its sides as an axis. Cylinders generated by similar rectangles revolved about corresponding sides as axes are called *similar cylinders of revolution*.



The figures represent in order from left to right an open cylindrical surface, an oblique circular cylinder, and a right circular cylinder, or a cylinder of revolution.

NOTE: The pupil should understand that all cylindrical surfaces are not necessarily closed and convex surfaces and that cylinders may have right sections other than circles. The theorems and exercises which are to be studied deal only with circular cylinders and where no misunderstanding may arise, the word *cylinder* may be taken to mean "a cylinder with circular bases."

152. Preliminary Theorems. Cylinders.

a) *The elements of a cylinder are equal and parallel.*

b) *Any two elements of a cylinder determine a plane.*

c) *Through a point in a cylindrical surface, there is only one element.*

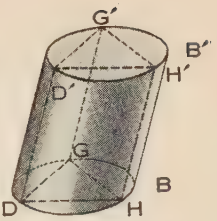
d) *A section of a cylinder formed by a plane which passes through an element is a parallelogram.*

e) *A section of a right cylinder formed by a plane which passes through an element is a rectangle.*

f) *Any element of a right cylinder equals the altitude.*

g) *In similar cylinders of revolution, the altitudes have the same ratio as the radii or the diameters of the bases.*

Proposition 1 (c n)



153. Theorem. *The bases of a cylinder are congruent.*

Given: The bases of a cylinder are B and B'.

To prove: $B \cong B'$.

Proof:

Statements	Facts
1. Let D and H be two fixed points and G any other point in the boundary of B and let DD', HH', and GG' be elements through those points. Draw HG, GD, DH, $H'G'$, $G'D'$, and $D'H'$.	1. § 152*c*.
2. $DHH'D'$, $HGG'H'$, $GDD'G'$ are ▱.	2. § 152*d*.
3. $DH = D'H'$, $HG = H'G'$, $GD = G'D'$.	3. Why?
4. $\triangle DHG \cong \triangle D'H'G'$.	4. Why?
5. If base B is placed on base B' so that D and H fall on D' and H' respectively, any other point (G) in the boundary of B will fall on some corresponding point (G') in the boundary of B' and B will coincide with B' and be congruent to it.	5. Def. of Congruence.

154. Corollary 1. *Parallel sections of a cylindrical surface are congruent.*

155. Corollary 2. *The line segment joining the centers of the bases of a circular cylinder passes through the centers of all parallel sections of the cylinder.*

156. Tangent Plane. A plane which passes through one and only one element of a cylinder is called a *tangent plane.*

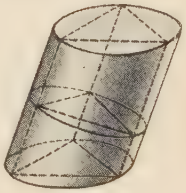
157. Circumscribed Prism. A prism whose lateral faces are tangent planes of a cylinder and whose bases are respectively coplanar with the bases of the cylinder is called a *circumscribed prism.* The cylinder is said to be inscribed in the prism.

158. Inscribed Prism. A prism whose lateral edges are elements of a cylinder and whose bases are coplanar with the bases of the cylinder is called an *inscribed prism.* The cylinder is said to be circumscribed about the prism.

159. Principle. *If a prism is inscribed in or circumscribed about a cylinder and if the number of the lateral faces is continuously increased without bound, the lateral area* (S), *the perimeter of a right section* (p), *the area of the base* (B), *and the volume* (V) *of the cylinder may be substituted in an equation for the corresponding values of either prism.*

Proposition 2 (c N)

160. Theorem. *The lateral area of a cylinder equals the product of an element and the perimeter of a right section.*



Given: A cylinder with its lateral area denoted by S, the perimeter of a right section by p, and an element by e.

To prove: $S = pe$.

Proof:

Statements	Facts
1. Let a prism be inscribed in the cylinder and let the number of its lateral faces increase without bound. Denote the lateral area and the perimeter of a right section of the prism in any instance by S' and p' respectively.	1. O 1, §§ 152*c*, 152*d*, 158.
2. For any such prism, $S' = p'e$.	2. Why?
3. The number of lateral faces increases without bound, $S' \to S$, $p' \to p$, therefore, $S = pe$.	3. § 159.

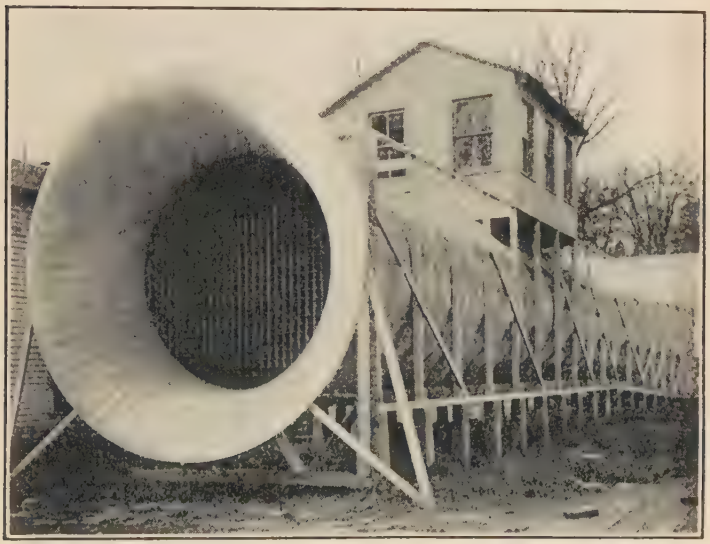
Question. Why is the word *perimeter* used in the theorem of § 160 rather than the word *circumference?*

161. Corollary 1. *The lateral area of a right circular cylinder (or of a cylinder of revolution) equals the product of an altitude and the circumference of a base.* ($S = hc$.)

162. Corollary 2 (c). *If the lateral area of a right circular cylinder is denoted by S, the altitude by h and the radius of the base by r, $S = 2\pi rh$.* [*The total area, $T = 2\pi r(h + r)$.*]

163. Corollary 3 (c). *The lateral areas and the total areas of similar cylinders of revolution have the same ratio as the squares of their altitudes or as the squares of the radii of their bases.*

HINT: $\frac{S}{S'} = \frac{2\pi rh}{2\pi r'h'}$; $\frac{r}{r'} = \frac{h}{h'} = \frac{r+h}{r'+h'}$; $\frac{T}{T'} = \frac{2\pi r(h+r)}{2\pi r'(h'+r')}$.



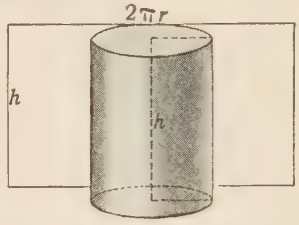
Courtesy Keystone View Co. Inc. of N. Y.

A CYLINDRICAL WIND TUNNEL

The picture shows the largest of three wind tunnels used by the Bureau of Standards at Washington, D. C., to determine the performance of structures and materials in wind streams. The forces acting on a model or other object being tested are measured by instruments located in the house shown above the tunnel. This tunnel is ten feet in diameter. If a wind stream of 75 mi. an hour is generated, how many cubic feet of air are emptied from the mouth of the tunnel in twenty minutes?

Exercises. Areas of Cylinders

1. If the lateral surface of a right circular cylinder is cut along one element and spread out flat, what figure is formed? The total area of a right circular cylinder is equal to the sum of the areas of what three figures?



2. Find the lateral area of a right circular cylinder if the altitude and the circumference of the base are respectively 3 and 4; 5 and 6; 10 and 15; 8 and 6π; 4 and 10π; h and c; $2\sqrt{6}$ and $\frac{\pi}{\sqrt{3}}$.

3. How much tin is required to make a can the radius of whose base is 2″ and whose height is 6″, if 10 sq. in. is allowed for seams.

4. What are the dimensions of a cylindrical tank whose lateral area is 4620 sq. in., if the height is $1\frac{3}{4}$ times the radius of the base? (Use $\pi = \frac{22}{7}$.)

5. In the formula $T = 2\pi r(h + r)$, solve for h; for r.

6. Draw a cylinder of revolution whose altitude equals the diameter of the base. Express S and T in terms of r; in terms of h.

7. For a cylinder of revolution, find the ratio of the area of the base (*a*) to the lateral area; (*b*) to the total area. (*c*) Find the ratio of S to T.

8. In a cylinder of revolution (*a*) does the area of the base vary directly as r; (*b*) does the lateral area; (*c*) does the total area? Express your answer in each case in the form of an equation.

9. In a cylinder of revolution, (*a*) does the area of the base vary directly as h? (*b*) does the lateral area? (*c*) does the total area? Express your answer in each case in the form of an equation.

10. In two similar cylinders of revolution, if the ratio of the altitudes is 9 : 4, find the ratio of (*a*) the radii of the bases; (*b*) the lateral areas; (*c*) the total areas.

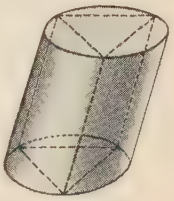
11. In two similar cylinders of revolution, if the ratio of the lateral areas is 9 : 4, find the ratio of the radii of the bases. What is the ratio of their total areas?

12. Find the total area of a right circular cylinder generated by a rectangle whose diagonal is 50 in. and the ratio of whose sides is 3 : 4. Are there two answers?

13. In two similar cylinders of revolution, if $S = 27$, $r = 3$, and $r_1 = 5$, find S_1. If $S = 192$, $S_1 = 27$, and $r = 8$, find r_1.

Proposition 3 (c N)

164. Theorem. *The volume of a circular cylinder equals the product of its base and altitude.*



Given: V is the volume of a circular cylinder with the altitude denoted by h and the base by B.

To prove: $V = Bh$.

Proof: STATEMENTS	FACTS
1. Let a prism be inscribed in the cylinder and let the number of its lateral faces increase without bound. Denote the area of the base and the volume of the prism in any instance by B' and V' respectively.	1. O 1, §§ 152*c*, 152*d*, 158.
2. For any such prism, $V' = B'h$.	2. Why?
3. The number of lateral faces increases without bound, $B' \to B$, $V' \to V$, therefore, $V = Bh$.	3. § 159.

165. Corollary 1 (c). *In a cylinder of revolution,* $V = \pi r^2 h$.

166. Corollary 2. *Let the volumes of two circular cylinders be denoted by* V *and* V_1, *their altitudes by* h *and* h_1, *and the radii of their bases by* r *and* r_1,

a) $V : V_1 = r^2 h : r_1^2 h_1$;

b) $V : V_1 = r^2 : r_1^2$, if $h = h_1$;

c) $V : V_1 = h : h_1$, if $r = r_1$;

d) $V = V_1$, if $h = h_1$ and $r = r_1$.

NOTE: The pupil should give each statement in full and prove each part.

167. Corollary 3. *The volumes of similar cylinders of revolution have the same ratio as the cubes of their altitudes or as the cubes of the radii of their bases.*

HINT: The proof follows the same plan as § 163.

Exercises. Volumes of Cylinders

1. Find the volume of a right circular cylinder, if the altitude and radius of the base are respectively (*a*) 4 and 3; (*b*) 3 and 4; (*c*) 5 and 6; (*d*) x and y; (*e*) $4\sqrt{3}$ and $\sqrt{2}$.

2. Does the volume of a cylinder of revolution vary directly as the altitude or the radius of the base?

3. How would you increase the volume of a right circular cylinder four times (*a*) by changing only the altitude? (*b*) by changing only the base? Prove your answer in each case.

4. Find the volume of a right circular cylinder in terms of S and π, if S is the lateral surface, and the altitude of the cylinder is twice the radius of the base.

5. Find the per cent of increase in the volume of a cylinder, if h is increased 25% and r is increased 50%.

6. If $h = 2r$ and $T = 600\pi$, find V of a right circular cylinder.

7. A 3000-gallon cylindrical oil tank is 5 ft. in diameter. What is its height?

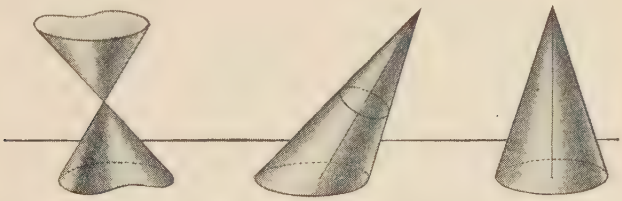
8. Given a cylinder and its inscribed and circumscribed square prisms. Find the continued ratio of their volumes.

168. Closed Conical Surface. The figure generated by a moving straight line that continuously intersects a fixed closed curve as the line passes through a fixed point non-coplanar with the curve is called a *closed conical surface.* The fixed point is called the *vertex;* the moving line in any of its positions is called an *element;* the two parts of the surface which have only the vertex in common are called *nappes.*

169. Cone. The solid formed by the portion of a nappe of a closed conical surface determined by and included between the vertex and a plane which cuts each element of the surface is called a *cone of one nappe,* or simply a *cone.* The vertex of the surface is also the *vertex* of the cone; the intersection of the plane and the surface is called the *base;* the perpendicular from the vertex to the plane of the base is called the *altitude;* the line joining the vertex to the center of the base is called the *axis;* the portion of an element of a

conical surface between the vertex and base of a cone is called an *element* of the cone; the portion of the conical surface between the vertex and the base is called the *lateral surface;* the area of the lateral surface is called the *lateral area;* the sum of the lateral area and the area of the base is called the *total area.*

170. Kinds of Cones. A cone whose base is a circle is called a *circular cone;* if the axis of a circular cone is perpendicular to the plane of the base, the cone is called a *right circular cone,* and if the axis is not perpendicular to the plane of the base, the cone is called an *oblique circular cone.* A right circular cone is also called a *cone of revolution* since it may be generated by revolving a right triangle about one of its legs as an axis. Cones generated by similar right triangles revolved about corresponding sides as axes are called *similar cones of revolution.*



The figures represent in order from left to right a cone of two nappes, an oblique circular cone, a right circular cone or a cone of revolution.

NOTE: The pupil should understand that all conical surfaces are not necessarily closed and convex surfaces and that cones may have other than circular bases. The theorems and exercises which are to be studied deal only with cones whose bases are circles and where no misunderstanding may arise, the word *cone* may be taken to mean "a cone with a circular base."

171. Preliminary Theorems. Cones.

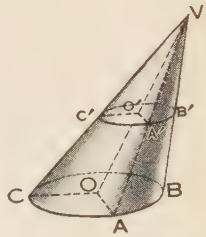
a) The elements of a right circular cone are equal.

b) The axis of a right circular cone is the altitude of the cone.

c) A section of a right circular cone formed by a plane which passes through an element is an isosceles triangle.

d) In similar cones of revolution, the altitudes have the same ratio as the radii or the diameters of the bases.

172. Slant Height of a Cone of Revolution. Any element of a cone of revolution is called the *slant height.*



Proposition 4 (C N)

173. Theorem. *A section of a cone parallel to the base is a circle.*

Given: $A'B'C'$ is a section of a cone parallel to the base ABC.

To prove: Section $A'B'C'$ is a circle.

Proof:

STATEMENTS	FACTS
1. Let VO be the axis of the cone intersecting the section in O' and let C' and A' be any two points in the boundary of $A'B'C'$. Let the planes determined by VO and points C' and A' intersect the planes of the base and of the section in OC, OA, $O'C'$, and $O'A'$ respectively.	1. Why?
2. $O'C' \parallel OC$, $O'A' \parallel OA$.	2. Why?
3. $\triangle VO'C' \sim \triangle VOC$, $\triangle VO'A' \sim \triangle VOA$.	3. Why?
4. $\frac{O'C'}{OC} = \left(\frac{VO'}{VO}\right) = \frac{O'A'}{OA}$.	4. Why?
5. $OC = OA$.	5. Why?
6. $O'C' = O'A'$.	6. Why?
7. Section $A'B'C'$ is a circle.	7. Why?

174. Corollary (N). *If a section of a cone is parallel to the base, (a) any element and the altitude are divided propor-*

tionally; (b) the area of the base and the area of the section have the same ratio as the square of the altitude and the square of the perpendicular from the vertex to the section.

175. Tangent Plane. A plane which passes through one and only one element of a cone is called a *tangent plane.*

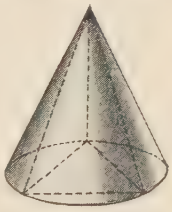
176. Circumscribed Pyramid. A pyramid whose lateral faces are tangent planes of a cone and whose base is coplanar with the base of the cone is called a *circumscribed pyramid.* The cone is said to be inscribed in the pyramid.

177. Inscribed Pyramid. A pyramid whose lateral edges are elements of a cone and whose base is coplanar with the base of the cone is called an *inscribed pyramid.* The cone is said to be circumscribed about the pyramid.

178. Principle. *If a pyramid is inscribed in or circumscribed about a cone and if the number of the lateral faces is continuously increased without bound, the lateral surface* (S)*, the circumference of the base* (c)*, the area of the base* (B)*, and the volume* (V) *of the cone may be substituted in an equation for the corresponding values of either pyramid; and if a regular pyramid is inscribed in a right circular cone, the slant height* (l) *of the cone may also be similarly substituted.*

Proposition 5 (c N)

179. Theorem. *The lateral area of a right circular cone equals one-half the product of its slant height and the circumference of its base.*



Given: A right circular cone with its lateral area denoted by S, the circumference of its base by c, and its slant height by l.

To prove: $S = \frac{1}{2}lc$.

Proof: STATEMENTS	FACTS
1. Let a pyramid be inscribed in the cone and let the number of its lateral faces increase without bound. Denote the lateral area, the slant height, and the perimeter of the base of the pyramid in any instance by S', l', and p' respectively.	1. § 177.
2. For any such pyramid, $S' = \frac{1}{2} l'p'$.	2. Why?
3. The number of lateral faces increases without bound, $S' \to S$, $l' \to l$, $p' \to c$, therefore, $S = \frac{1}{2} lc$.	3. § 178.

180. Corollary 1 (c). *If the lateral area of a right circular cone is denoted by S, the slant height by l, and the radius of the base by r, $S = \pi rl$.* [The total area $T = \pi r\,(r + l)$.]

181. Corollary 2 (c). *The lateral areas and the total areas of similar cones of revolution have the same ratio as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases.*

HINT: The proof follows the same plan as § 163.

Exercises. Areas of Cones

1. If a right circular cone were cut along one of its elements and then flattened out, what plane figure would be formed by its lateral surface? Write as many geometric formulas as you can which have the form $A = \frac{1}{2} bc$.

2. If a plane sector whose radius is 20 and whose central angle is 60° is rolled to form a conical surface, find the area of its lateral surface. Find the lateral area, if the angle of the sector is 45°, 90°, 120°, 180°.

3. A right triangle is revolved about its shorter leg as an axis. Find the total area generated, if the legs are (*a*) 6 and 8; (*b*) 12 and 16; (*c*) 8 and 15; (*d*) n and $2n$; (*e*) a and $a\sqrt{3}$.

4. Find the total area for each case of Ex. 3, if the longer leg is used as an axis.

5. Find the slant height of a right circular cone, if the radius of the base is 6 and the total area is (*a*) 84π; (*b*) 108π; (*c*) 132π.

6. Find the radius of the base of a right circular cone, if the slant height is 6 and the total area is (*a*) 40π; (*b*) 55π; (*c*) 27π.

7. Find the total area of a right circular cone, if the radius of the base is 10 and if the angle formed between a radius and the slant height equals (*a*) 30°; (*b*) 45°; (*c*) 60°; (*d*) 70°.

8. A right triangle is revolved about its hypotenuse as an axis. Find the area of the total surface generated, if the legs of the right triangle are (*a*) 6 and 8; (*b*) 8 and 15; (*c*) 9 and 12.

9. A right triangle whose legs are 5 and 12 is revolved about (*a*) side 5 as an axis, (*b*) side 12 as an axis, (*c*) about the hypotenuse as an axis. Find the ratio of the total areas of the figures generated.

10. Given T, the total area, and l, the slant height, of a cone of revolution. Find r, the radius of the base, in terms of T and l.

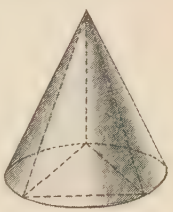
11. Find the total area of an equilateral cone, the diameter of whose base is equal to the slant height of the cone.

12. A, B, C, and D are similar cones of revolution, whose slant heights are 6, 8, 10, and 12. Find the ratio of their total areas.

13. The ratio of the lateral areas of three similar cones of revolution is 1 : 2 : 3. Find the total area of each cone, if the altitude of the smallest cone is n and the radius of its base is $2\,n$.

Proposition 6 (c N)

182. Theorem. *The volume of a cone equals one-third the product of its altitude and base.*



Given: A cone with its volume denoted by V, the altitude by h, and the area of the base by B.

To prove: $V = \frac{1}{3}\,Bh$.

(The pupil may complete the demonstration.)

183. Corollary 1 (c). *In a cone of revolution,* $V = \frac{1}{3}\,\pi r^2 h$.

184. Corollary 2. *Let the volumes of two right circular cones be denoted by* V *and* V_1, *their altitudes by* h *and* h_1, *and the radii of their bases by* r *and* r_1,

a) $V : V_1 = r^2\,h : r_1^2\,h_1$;

b) $V : V_1 = r^2 : r_1^2$, if $h = h_1$;

c) $V : V_1 = h : h_1$, if $r = r_1$;

d) $V = V_1$, if $h = h_1$ and $r = r_1$.

NOTE: The pupil should give each statement in full and prove each part.

185. Corollary 3 (c). *The volumes of similar right circular cones have the same ratio as the cubes of their altitudes, or as the cubes of their slant heights, or as the cubes of the radii of their bases.*

Exercises. Volumes of Cones

1. Given $V = \frac{1}{3}\pi r^2 h$, solve for h; solve for r.

2. A circular cylinder and a right circular cone have equal bases and equal volumes. What is the ratio of their altitudes? If they have equal altitudes and equal volumes, what is the ratio of the radii of their bases?

3. Find the volume of a right circular cone, if the radius of the base (r) and the slant height (l) are respectively (a) 3 and 5; (b) 4 and 5; (c) 8 and 17; (d) 15 and 17; (e) a and $2a$; (f) $a\sqrt{2}$ and $a\sqrt{3}$.

4. Find the volume of a right circular cone, if the altitude (h) and the slant height (l) are respectively (a) 7 and 25; (b) 24 and 25; (c) 9 and 15; (d) $a\sqrt{3}$ and $2a$.

5. Find the volume generated by the revolution of a right triangle whose legs are 6 and 8 (a) about the shorter leg as an axis; (b) about the longer leg as an axis; (c) about the hypotenuse as an axis.

6. In two similar cones of revolution, the radii of the bases are respectively 6 and 8. What is the ratio (a) of their altitudes, (b) of their volumes, (c) of their slant heights, (d) of their lateral areas, (e) of their total areas? If the altitude of the smaller cone is 8, find the altitude, the volume, the slant height, the total area of the larger cone.

186. Frustum of a Cone. The terms frustum, base, altitude, slant height, lateral area, etc., as applied to a frustum of a regular pyramid have the same meaning when applied to a frustum of a right circular cone.

187. Inscribed and Circumscribed Frustums. If a right circular cone has a regular pyramid inscribed in it and a regular pyramid circumscribed about it, a plane parallel to the base forms two frustums of pyramids, the one inscribed in and the other circumscribed about the frustum of the cone.

188. Principle. *If a frustum of a regular pyramid is inscribed in or circumscribed about a frustum of a right circular cone and if the number of the lateral faces is continuously increased without bound, the slant height, the circumferences*

and areas of the bases, the lateral area, and the volume of the frustum of the cone may be substituted in an equation for the corresponding values of either frustum of a pyramid.

189. Theorem (c n). *The lateral area of a frustum of a right circular cone equals one-half the product of its slant height and the sum of the circumferences of its bases.* $[S = \frac{1}{2} l(c + c').]$

190. Corollary (c). *In a frustum of a right circular cone,* $S = \pi l(r + r')$.

191. Theorem (n). *The volume of a frustum of a right circular cone equals one-third the product of its altitude and the sum of the bases and their mean proportional.*

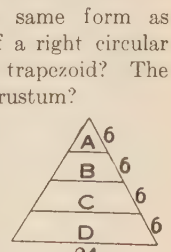
$$[V = \tfrac{1}{3} h(B + B' + \sqrt{BB'}).]$$

192. Corollary (n). *In a frustum of a right circular cone,* $V = \frac{1}{3} \pi h(r^2 + r'^2 + rr')$.

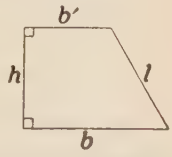
Exercises. Areas and Volumes of Frustums of Cones

1. What formula of plane geometry has the same form as $S = \frac{1}{2} l(c + c')$? May the lateral area of a frustum of a right circular cone be found by using the formula for the area of a trapezoid? The altitude of the trapezoid corresponds to what for the frustum?

2. The annexed figure represents a section of a right circular cone formed by a plane through the altitude. B, C, and D are cross sections of frustums formed by parallel planes which divide the slant height into four equal parts of 6 units each. If the diameter of the base is 24, find the total area and volume of D; of $C + D$; of C; of $B + C + D$; of B; of A.



3. The annexed figure is a trapezoid whose bases b and b' are perpendicular to h. Find the total area and volume of the figure generated by revolving the trapezoid about h as an axis, if b, b', and h are respectively (*a*) 12, 9, 6; (*b*) 16, 12, 6; (*c*) 18, 15, 5.

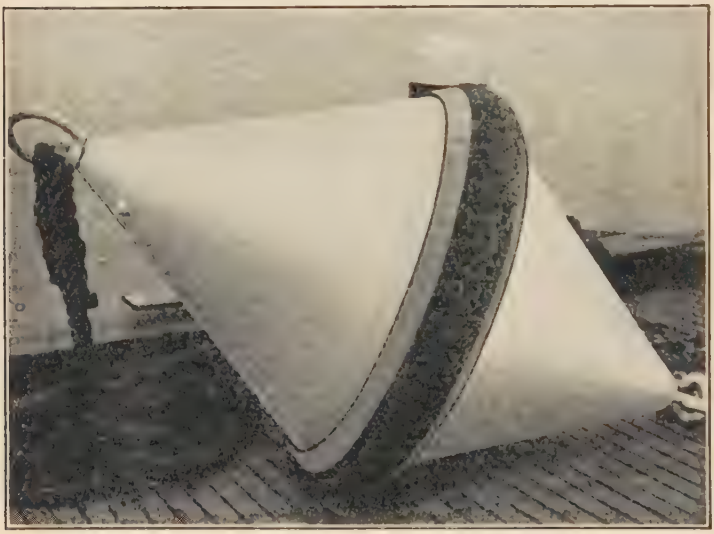


4. Show how the formula, $V = \frac{1}{3} \pi h(r^2 + r'^2 + rr')$ for the volume of a right circular cone may be obtained from the formula for the volume of a prismatoid, $V = \frac{1}{6} h(B + B' + 4M)$.

HINT: Let $B = \pi r^2$, $B' = \pi r'^2$; then $4M = 4\pi \left(\frac{r + r'}{2}\right)^2$.

5. A right circular cone, whose altitude is denoted by h and the radius of the base by r, is divided into three equal parts by two planes parallel to the base. Find the altitude of each of the three parts and the radius of each of the circular sections.

6. The radii of the bases and the slant height of a frustum of a right circular cone are respectively a, $2a$, and $3a$. Find the total area and volume of the frustum in terms of a and π.



Courtesy Keystone View Co. Inc. of N. Y.

A DOUBLE CONICAL BUOY

The buoy shown in the picture is essentially a double cone of sheet iron. The largest diameter of the buoy is 3 ft. 4 in. and the length from the tip of one cone to the tip of the other is 5 ft. 6 in. Find the weight of the buoy, if the sheet iron used is $\frac{1}{4}$ in. thick and if one cubic inch of iron weighs one-quarter of a pound. What is the weight of the water displaced when the buoy is completely submerged, if one cubic foot of water weighs sixty-three pounds?

Exercises. Equal Volumes

1. In two equal right circular cylinders, the altitude and radius of one are 20 and 12 respectively. If the altitude of the other is 5, what is the radius?

2. A right circular cylinder and a right circular cone have equal volumes. The radius of the base of each is 12. If the altitude of the cylinder is 6, what is the altitude of the cone?

3. If a cylindrical bar 20″ long and 4″ in diameter is drawn out into a rod $\frac{1}{2}$″ in diameter, how long is the rod?

4. How many cylinders 6″ long and $2\frac{1}{2}$″ in diameter are equal in volume to a cylinder 12″ long and 6″ in diameter?

5. A pail in the form of a frustum of a right circular cone is 10″ deep and its lower and upper bases are respectively 10″ and 14″ in diameter. How many cylindrical glass jars 8″ high and 4″ in diameter can be filled with the contents of the pail, when it is filled within two inches of the top?

6. A regular pyramid, a right circular cone, and a circular cylinder have equal volumes. Find the ratio of the areas of their bases, if their altitudes have the ratio of 1 : 2 : 3.

Summary. Properties of Similar Figures

It will be helpful to the pupil to note that in all *similar* figures of plane and solid geometry the following relations exist between any corresponding lines, areas, or volumes:

a) corresponding lines have the same ratio as any other corresponding lines;

b) areas have the same ratio as the *squares* of any two corresponding lines or as the square of their ratio;

c) volumes have the same ratio as the *cubes* of any two corresponding lines or as the cube of their ratio;

d) corresponding lines have the same ratio as the *square roots* of corresponding areas or as the *cube roots* of corresponding volumes.

These relations are true also for circles, similar figures in connection with circles, spheres, and similar figures in connection with spheres. The pupil should check these properties in the propositions or formulas of the text.

Miscellaneous Propositions

1. If a plane through an element of a cylinder intersects its surface in a point not in that element, the intersection of the plane and the cylinder is a parallelogram.

2. A plane determined by an element of a cylinder and a tangent to its base is tangent to the cylinder.

3. Sections of a circular cylinder parallel to its bases are equal circles and the straight line joining the centers of the bases passes through the centers of all such sections

4. If two circular cones have equal bases and equal altitudes, sections parallel to the bases and at equal distances from the vertices are equal.

5. A plane determined by an element of a circular cone and a tangent to its base is tangent to the cone.

6. The circumferences of parallel sections of a cone have the same ratio as the distances of the sections from the vertex; and the areas of the sections have the same ratio as the squares of those distances.

7. The axis of a right circular cone passes through the center of every section parallel to the base.

Review Oral Exercises

1. What is the altitude of a cylinder? May a cylinder have more than one altitude?

2. How is a right section of a cylinder formed?

3. Name four kinds of cylinders.

4. Is every section of a cylinder a circle? Under what conditions is it a circle?

5. Are all cylinders similar? Under what conditions are two cylinders similar?

6. What is the locus of points in space d units from a given line?

7. For what cylinders are all the elements equal to the altitude?

8. May a cylinder be inscribed in and circumscribed about any prism?

9. Compare the formula for the lateral area of a prism and the formula for the lateral area of a cylinder.

10. State the formulas for the lateral area and volume of a cone of revolution.

11. May the lateral area and volume of a cone be derived from the corresponding formulas for a cylinder by considering one of its bases to equal zero?

12. Do all cones have a slant height?

13. To what two solids does the formula, $V = Bh$ apply?

14. To what two solids does $V = \frac{1}{3} Bh$ apply?

15. To what two solids does $V = \frac{1}{3} h(B + B' + \sqrt{BB'})$ apply?

16. What figure of plane geometry is equivalent to the lateral surface of (*a*) a right circular cone, (*b*) a right circular cylinder, (*c*) a frustum of a right circular cone?

17. Translate the following formulas into general rules: (*a*) $S = 2\pi rh$; (*b*) $V = \pi r^2 h$; (*c*) $S = \frac{1}{2} lc$; (*d*) $S : S' = r^2 : r'^2$; (*e*) $V = \frac{1}{3} \pi r^2 h$; (*f*) $S = \frac{1}{2} l(c + c')$.

18. What figure is formed by rotating a right triangle about its hypotenuse as an axis?

19. What figure is formed by rotating an isosceles trapezoid about the line joining the mid-points of its bases as an axis?

20. How many times is the volume of a right circular cone increased, (*a*) if the altitude is doubled? (*b*) if the radius of the base is doubled? (*c*) if the altitude and radius are both doubled?

21. In two similar cylinders or cones of revolution, if the ratio of the altitudes is 1 : 8, what is the ratio of the lateral areas? What is the ratio of the volumes?

22. Under equal pressures an 8″ water main should discharge how many times as much water in a given period as a 2″ pipe?

23. The angle formed by the slant height and altitude of a cone is 30°. If the slant height of the cone is 10″, find the radius of the base. What is the lateral area of the cone?

Miscellaneous Exercises

1. The total area of a right circular cylinder is 100π and the altitude equals the radius of the base. Find the dimensions of the cylinder.

2. How many gallons of gasoline are contained in a cylindrical automobile tank 12 in. in diameter and 33 in. long?

3. Show that the volume of a right circular cylinder equals the product of its lateral area and half the radius of the base.

4. Find the ratio of the volumes of the cones circumscribed about and inscribed in a regular pyramid with a square base 6 in. on each side and with an altitude of 8 in.

5. Find the ratio of the lateral areas of the cones in Ex. 4.

6. Find the ratio of the volumes of the cylinders circumscribed about and inscribed in a 6-in. cube.

7. Find the ratio of the total areas of the cylinders in Ex. 6.

8. The sides of a square are 10 in. Find the total area and volume of the figure generated by revolving the square about its diagonal as an axis.

9. The area of the base of a right circular cone is 25π and its lateral area is 65π. Find its dimensions.

10. The area of the base of a right circular cone is 25π and its volume is 100π. Find its dimensions.

11. How many gallons of water pour in one hour from a $\frac{3}{4}''$ pipe, if the rate of flow is 100 ft. a minute?

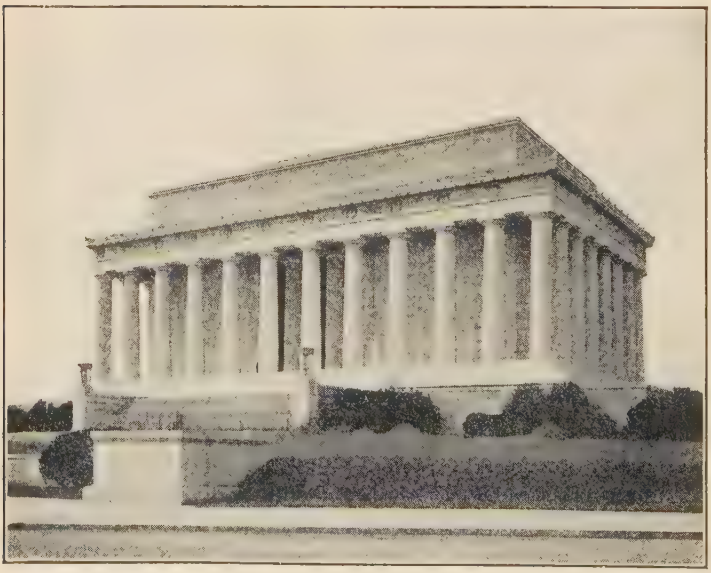
12. A drinking glass is $3\frac{3}{4}$ in. deep with a lower diameter of $2\frac{1}{4}$ in. and an upper diameter of $2\frac{3}{4}$ in. If the glass is filled to a depth of 3 in., what decimal part of a quart does it contain? How many such glasses could be completely filled with one quart?

13. The tank of an oil-tank car is 25 ft. long and 6 ft. in diameter. How many gallons of oil does it contain, if filled to a depth of $4\frac{1}{2}$ ft.?

14. A rectangular hexagon $ABCDEF$ whose sides are each 6 in. revolves about diagonal AD as an axis. Find the total area and volume of the figure generated.

15. In a frustum of a right circular cone h, r, and r_1 are respectively 8, 15, and 10. Find its volume from the difference of two cones. Check your result using the usual formula.

16. The 36 columns of the Lincoln Memorial at Washington, D. C., are the largest of their kind in the world. Each column is 7 ft. in diameter and 44 ft. high. Find their total volume.



THE LINCOLN MEMORIAL
(*See Ex. 16*)

CHAPTER FOUR

THE SPHERE

193. Sphere. The closed surface generated by the revolution of a semicircle about its diameter as an axis is called a *sphere.* All points on the surface are equidistant from a point called the *center,* which is also the center of the generating semicircle. The terms *radius, chord, diameter,* and *tangent line* of a sphere are defined in the same manner as they were defined for a circle and as with circles, spheres are usually named by naming their centers. Half a sphere is called a *hemisphere.*

194. Preliminary Theorems. Spheres.

a) *Spheres with equal radii or equal diameters are equal* (*or congruent*).

b) *Radii or diameters of a sphere* (*or of equal spheres*) *are equal.*

c) *A point is within, on, or without a sphere according as its distance from the center is less than, equal to, or greater than a radius.*

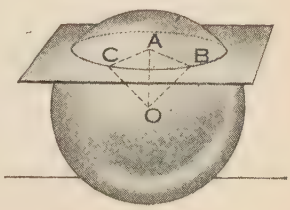
NOTE: Since the curvature of any portion of a sphere is the same as for any other portion of the same sphere, due to the constant radius from a fixed center, it will be assumed that the Postulate of Mobility holds true on a sphere.

Proposition 1 (C n)

195. Theorem. *A section of a sphere formed by a plane is a circle.*

Given: A section of sphere O made by a plane. B and C are any two points on the section.

To prove: The section is a circle.



Proof: STATEMENTS	FACTS
1. Let OA be perpendicular to the plane of the section and draw AB, AC, OC, and OB.	1. Why?
2. $OA \perp AB$, $OA \perp AC$.	2. Why?
3. $OB = OC$.	3. § 194*b*.
4. $AB = AC$.	4. § 21.
5. The section is a circle.	5. Why?

196. Circles of a Sphere. Lines which lie on a sphere are curved lines and they may be circles or arcs of circles of two kinds. A circle whose plane passes through the center of a sphere is called a *great circle;* a circle whose plane does not pass through the center is called a *small circle* of the sphere. The center of the sphere is the center of all great circles of that sphere. The diameter of a sphere which is perpendicular to the plane of a circle on the sphere is called the *axis* of that circle and the ends of the diameter are called the *poles* of that circle. The length of the smaller arc of a great circle which joins two points on a sphere is called the *spherical distance* between the points. The spherical distance from a point on a circle to its nearer pole is called the *polar distance* of the circle. One-fourth of a great circle is called a *quadrant.*

NOTE: The pupil should understand that in spherical geometry all figures on a sphere are composed of circles or of arcs of circles and that such lines correspond to straight lines in plane geometry. It can be shown, moreover, that the length of an arc of a great circle which connects two points on a sphere is shorter than the length of any other arc on the sphere between those points. (See p. 136 of the Supplement.) It is also possible to prove that two points (except end points of a diameter) on a sphere determine one and only one great circle. Because of these resemblances to the properties of straight lines in plane geometry, arcs of great circles on a sphere in spherical geometry are sometimes called "the straightest-straight lines on a sphere." Considerable attention is given, therefore, to the properties of great circles in spherical measurements.

197. Preliminary Theorems. Circles of Spheres.

a) *The axis of a circle of a sphere passes through the center of the circle, and conversely.*

b) *Great circles of a sphere (or of equal spheres) are equal.*

c) *Great circles of a sphere bisect each other and bisect the sphere.*

d) (c) *Two points on a sphere, not the extremities of a diameter, determine one great circle and only one.*

HINT: The points and the center of the sphere determine a plane which intersects the sphere in a great circle.

e) (c) *Three points on a sphere determine one circle and only one.*

f) *Circles of a sphere whose planes are parallel have the same axis and the same poles.*

g) *Sections of a sphere (or of equal spheres) which are equidistant from the center are equal.*

h) *The intersection of the planes of any two great circles of a sphere is a diameter of the sphere.*

Question: What are the statements of plane geometry corresponding to (*d*) and (*e*)?

Oral Exercises. Circles of a Sphere

1. How many circles are there through two points on a sphere, which are the extremities of a diameter?

2. How many circles are there through two points (not end points of a diameter) on a sphere? how many great circles?

3. What kind of circle of the earth is (*a*) the equator? (*b*) a meridian? (*c*) a parallel of latitude?

4. The north pole of the earth is approximately the pole of what circle or circles? Why is the term "approximately" used? Is the south pole the pole of the same circles as the north pole?

5. Into how many parts do three great circles divide a sphere, if they do not have the same common diameter?

6. How many circles can be drawn on a hemisphere, if they are (*a*) great circles? (*b*) small circles?

7. If one ship is in 40° N. Lat. and a second vessel is in 40° S. Lat., are they respectively north and south of one another?

8. If two lines on the earth's surface were run east and west, would they intersect? If two lines were run north and south, would they intersect? If so, at what points? Are lines which run due north and south parallel?

9. If a great circle is on a sphere and if a point on the sphere is non-collinear with the circle, is it possible to have a second great circle through the given point and parallel to the given great circle?

10. Why are all great circles of a sphere equal?

11. The circumference of a great circle of the earth is about 25,000 miles. What is the approximate length of a degree of a great circle of the earth? Is a degree of every circle of the earth the same length?

12. How many great circles of a sphere have the same pole? How many small circles have the same pole? May a pole of a great circle be the pole of a small circle? Why?

13. What is the locus of the centers of the circles on a sphere formed by a number of parallel planes intersecting the sphere?

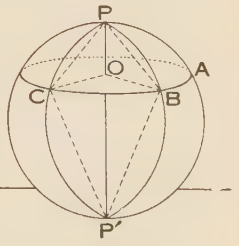
14. The planes of two great circles of a sphere intersect in what line?

15. How many circles can be passed through three points on a sphere? Is such a circle a great circle?

16. If the planes of two small circles of a sphere are equidistant from the center of the sphere, why are the radii of the small circles equal?

Proposition 2 (c N)

198. Theorem. *The spherical distances of the points on a circle of a sphere from either pole are equal.*



Given: P and P' are the poles of circle ABC of a sphere; B and C are any two points on the circle; PCP' and PBP' are arcs of great circles.

To prove: $\overset{\frown}{PC} = \overset{\frown}{PB}$ and $\overset{\frown}{P'C} = \overset{\frown}{P'B}$.

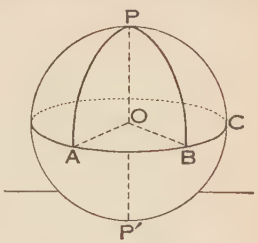
Proof: STATEMENTS	FACTS
1. PP' is perpendicular to the plane of ABC and passes through the center O.	1. §§ 196, 197.
2. Connect B and C with P, O, and P'.	2. A 27.
3. $OB = OC$.	3. Why?
4. $PB = PC$, $P'B = P'C$.	4. § 21.
5. The great circles PBP' and PCP' are equal circles.	5. § 197*b*.
6. $\overset{\frown}{PB} = \overset{\frown}{PC}$, $\overset{\frown}{P'B} = \overset{\frown}{P'C}$.	6. Why?

199. Corollary 1. *The polar distance of a great circle is a quadrant.*

200. Corollary 2. *Straight line segments which join points of a circle on a sphere to the same pole are equal.*

Proposition 3 (C N)

201. Theorem. *A point on a sphere at a quadrant's distance from each of two points, which are not the extremities of the same diameter, is a pole of the great circle through the two points.*



Given: P is a point on a sphere such that $\overset{\frown}{PA} = \overset{\frown}{PB} = 90°$ and ABC is a great circle whose center is O.

To prove: P is a pole of circle ABC.

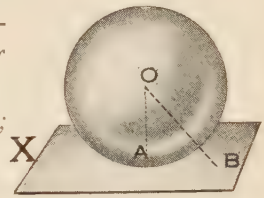
Proof: STATEMENTS	FACTS
1. Connect O with P, A, and B.	1. A 27.
2. $\overset{\frown}{PA} = \overset{\frown}{PB} = 90°$.	2. Why?
3. $\angle POA = \angle POB = 90°$.	3. J 2.
4. $POP' \perp$ plane ABC.	4. § 13.
5. P is a pole of circle ABC.	5. § 196.

202. Tangent Lines, Planes, and Spheres. A line and a sphere, a plane and a sphere, or two spheres which have one and only one point in common are said to be *tangent*

to each other. The common point is called the *point of tangency* or the *point of contact.*

Proposition 4 (c N)

203. Theorem. *A plane perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.*



Given: OA is a radius of sphere O; $X \perp OA$ at A.

To prove: X is tangent to sphere O.

Proof:

STATEMENTS	FACTS
1. Join O with any point (B) in X other than A.	1. Why?
2. $OB > OA$.	2. § 23.
3. B is without the sphere.	3. Why?
4. X is tangent to sphere O.	4. Why?

204. Corollary (c). *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

Exercises. Loci of Centers of Spheres

1. What is the locus of points at a given distance from a given point? from a given line? from a given plane?

2. What is the locus of points in space equidistant from two points? from three points?

What is the locus of the centers of spheres

3. passing through a given point and having a given radius?

4. passing through two given points?

5. passing through two given points and having a given radius?

6. passing through three given points?

7. passing through the vertices of a rectangle?

8. tangent to a given plane and having a given radius?

9. tangent to two parallel planes?

10. tangent to two perpendicular planes?

11. tangent to the faces of a dihedral angle?

12. tangent to the faces of a trihedral angle?

13. tangent to a given plane at a given point?

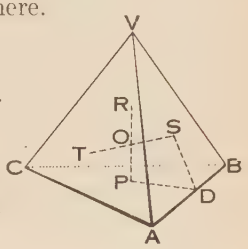
14. tangent to a given sphere and having a given radius?

15. tangent to two intersecting lines at their common point?

205. Inscribed and Circumscribed Spheres. If a sphere is tangent to each face of a polyhedron, it is said to be *inscribed* in the polyhedron; if all the vertices of a polyhedron lie on a sphere, the sphere is said to be *circumscribed* about the polyhedron. The polyhedron in the first case is said to be circumscribed about the sphere; in the second case, it is said to be inscribed in the sphere.

Proposition 5 (C N)

206. Theorem. *A sphere can be circumscribed about a tetrahedron.*



Given: $VABC$ is any tetrahedron.

To prove: A sphere can be circumscribed about $VABC$.

Proof:

Statements	Facts
1. Let P and S be the centers of the circles circumscribed about $\triangle ABC$ and $\triangle VAB$ respectively and let PR and ST be $\perp$ to the planes of those $\triangle$	1. E 20, § 18.
2. Join P and S with D, the mid-point of AB.	2. Why?
3. $PD \perp AB$ and $SD \perp AB$.	3. E 11.
4. Plane $PDS \perp AB$.	4. § 13.
5. Plane $PDS \perp$ plane ABC.	5. § 65.
6. PR lies in plane PDS; in like manner, ST also lies in plane PDS.	6. § 63.
7. $PR \perp PD$ and $ST \perp SD$.	7. § 12.
8. PR and ST intersect in a point (O).	8. E 26.
9. Since O is in PR, $OA = OB = OC$; since O is in ST, $OA = OB = OV$.	9. § 24.
10. $OA = OB = OC = OV$ and A, B, C, V lie on a sphere whose center is O and whose radius is OA.	10. Sub., § 205.

NOTE: Since O is the only point such that $OA = OB = OC = OD$, there is only one sphere which circumscribes $VABC$.

Oral Exercises

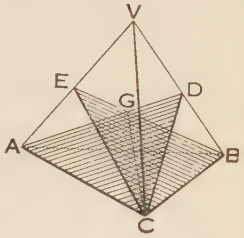
1. In the annexed figure of tetrahedron $VABC$, plane ACD bisects the dihedral angle V–AC–B. Of what points is plane ACD the locus?

2. If plane BCE bisects the dihedral angle V–BC–A, what is true of all points in the plane?

3. Planes ACD and BCE have which vertex of the tetrahedron as a common point? Do these planes then have two points in common? do they have a line in common?

4. Name the intersection of planes ACD and BCE. Describe the position of point G.

5. Are all points in CG equidistant from faces VCB and VCA? Why?



6. If the plane which bisects dihedral angle B–VC–A were drawn, would it intersect, pass through, or be parallel to CG?

7. If the plane which bisects dihedral angle C–VB–A were drawn, would it intersect, pass through, or be parallel to CG?

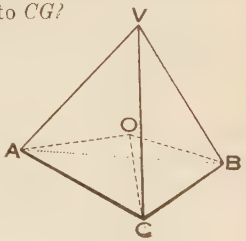
8. If the plane which bisects dihedral angle V–AB–C were drawn, would it intersect, pass through, or be parallel to CG?

Proposition 6 (c N)

207. Theorem. *A sphere can be inscribed in a tetrahedron.*

Given: $VABC$ is any tetrahedron.

To prove: A sphere can be inscribed in $VABC$.



Proof:

Statements	Facts
1. Let planes OAC, OCB, and OBA bisect the dihedral angles whose edges are AC, CB, and BA respectively.	1. § 61.
2. Planes OAC and OCB intersect in OC.	2. §§ 6, 8.
3. Plane OBA intersects OC in O.	3. § 2.
4. O is equidistant from the four faces of the tetrahedron, since it is the common intersection of the three bisecting planes.	4. § 71.

5. A sphere whose center is O and whose radius is the perpendicular from O to one of the faces is tangent to each of the four faces and is inscribed in the tetrahedron.	5. §§ 193, 203, 205.

NOTE: Since O is the only point equidistant from the four faces of the tetrahedron, there is only one sphere which is inscribed in $VABC$.

HISTORICAL NOTE. One of the mathematicians whose name is most closely associated with investigations on the sphere and other geometrical solids is Archimedes, who lived from 287 B.C. to 212 B.C. He wrote many short works on a great variety of subjects dealing principally with physics and mathematics. One of his treatises, called "Sphere and Cylinder," contained sixty propositions arranged in two books. In this work were expressions for the surface and volume of a pyramid, a cone, a cylinder, and a sphere as well as certain important ratios between them. Some idea of how Archimedes thought about these matters may be understood from the following translation from one of his works which is found in D. E. Smith's *History of Mathematics*. It reads as follows: "After I had thus perceived that a sphere is four times as large as the cone whose base is the greatest circle of the sphere and whose altitude is equal to the radius, it occurred to me that the surface of a sphere is four times as great as its largest circle. I proceeded from the idea that just as a circle is equal to a triangle whose base is the periphery (circumference) of the circle and whose altitude is equal to the radius, so a sphere is equal to a cone whose base is the same as the surface of the sphere and whose altitude is equal to the radius of the sphere."

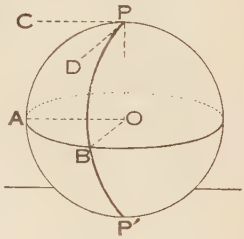
It is said that the Romans erected a monument over the grave of Archimedes at Syracuse, Sicily, on which was engraved the figure of a sphere inscribed in a cylinder.

208. Spherical Angles. The figure formed by two minor arcs of great circles of a sphere which have two end points in common is called a *spherical angle*. The common point is called the *vertex* and the arcs are called *sides* of the spherical angle.

209. Principle. *A spherical angle equals numerically the plane angle formed by the tangents to the arcs at the vertex of the angle.*

Proposition 7 (C)

210. Theorem. *A spherical angle equals numerically the arc of a great circle having the vertex of the angle as a pole and included between the sides of the angle (produced if necessary).*



Given: On the sphere whose center is O, spherical $\angle APB$ is formed by arcs of great circles ($P'AP$ and $P'BP$) which include another great circle arc AB whose pole is P.

To prove: Spherical $\angle APB$ = numerically $\overset{\frown}{AB}$.

Proof:

Statements	Facts
1. O is the center of the sphere and of all great circles on the sphere.	1. Given, § 196.
2. Join O with P, A, and B.	2. Why?
3. In planes PAP' and PBP', draw CP and DP each $\perp PO$.	3. Why?
4. $\overset{\frown}{PA} = \overset{\frown}{PB} = 90°$.	4. §§ 199, 196.
5. In planes PAP' and PBP', AO and BO are each $\perp PO$.	5. Why?
6. $CP \parallel AO$ and $DP \parallel BO$.	6. Why?
7. $\angle CPD = \angle AOB$.	7. Why?
8. CP and DP are tangents to $\overset{\frown}{P'AP}$ and $\overset{\frown}{P'BP}$ respectively.	8. Why?
9. $\angle AOB$ = numerically $\overset{\frown}{AB}$.	9. Why?
10. Spherical $\angle APB$ = numerically $\angle CPD$.	10. § 209.
11. Spherical $\angle APB$ = numerically $\overset{\frown}{AB}$.	11. Sub.

211. Corollary. *A spherical angle equals numerically the plane angle of the dihedral angle formed by the planes of its sides.*

Exercises. Spheres

1. If a spherical angle intercepts an arc of 60° on the great circle drawn with its vertex as a pole, (*a*) how many degrees are in the angle? (*b*) how many degrees are in the corresponding dihedral angle?

2. If the radius of a sphere is 13 in., what is the radius of a small circle of the sphere formed by a plane 5 in. from the center?

3. If the distances of a point on a sphere from the ends of a diameter are respectively 12 in. and 16 in., find the radius of the sphere.

4. The radii of two concentric spheres are 9 in. and 15 in. Find the area of the circle formed by the intersection of the larger sphere and a plane tangent to the smaller sphere.

5. How far from the center of a sphere whose radius is 10 in. is a plane which cuts the sphere in a small circle whose radius is 6 in.?

6. If a sphere is inscribed in a cube whose edge is 12 in., what is the radius of the sphere?

7. If a cube is inscribed in a sphere whose radius is 12 in., find the length of an edge of the cube.

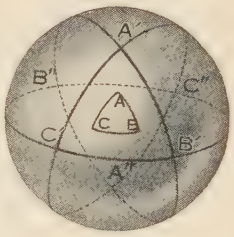
212. Spherical Polygon. A closed line on the surface of a sphere composed of three or more arcs of great circles is called a *spherical polygon.* The arcs are called *sides* of the polygon and their points of intersection are called *vertices.* The spherical angles formed are called the *angles* of the spherical polygon.

Such terms as *altitude, bisector of an angle, diagonal, median,* etc., when used in connection with spherical polygons refer to arcs of great circles which have the same relation to the spherical polygons as like-named straight lines have to plane polygons.

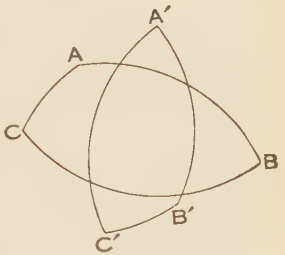
A spherical polygon of three sides is called a *spherical triangle.* Spherical triangles may be *right, isosceles, equilateral,* etc. The definitions of these terms correspond to those of plane triangles.

213. Polar Triangle. Three great circles of a sphere whose poles are the vertices of a given spherical triangle intersect forming eight spherical triangles. It is customary

to designate one of them as *the polar triangle* of the given triangle. For example, $A'B'$, $B'C'$, and $A'C'$ are arcs of great circles of which C, A, and B are the respective poles. Spherical triangle $A'B'C'$ is called the polar triangle of $\triangle ABC$ because it is the only triangle whose vertices lie on the same side of the arcs of the given triangle as the corresponding vertices of the given triangle, and none of its sides is greater than 180°.



NOTE: A given spherical triangle and its polar triangle are usually shown as one triangle entirely within the other. The pupil should understand, however, that it is not only possible to have the two triangles overlap, but also that such triangles are quite common. The annexed figure shows a spherical triangle ABC and its polar triangle $A'B'C'$. The unusual shape of the figure is due to the fact that arcs of great circles of a sphere have been represented by plane curves on a flat surface.



Oral Exercises

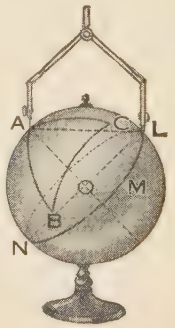
1. In the figure of § 213, name four spherical triangles.

2. In the figure of the preceding Note, $B'C'$ has what point as a pole? $A'C'$ has what pole? point C is the pole of what arc?

3. Has every spherical triangle a distinct polar triangle? Why?

4. If one vertex of a spherical triangle on the earth is taken at the south pole, one side of the polar triangle lies in what line?

5. If one vertex of a spherical triangle on the earth is taken at the north pole and one side lies on the equator, describe the position of one side and one vertex of the polar triangle.



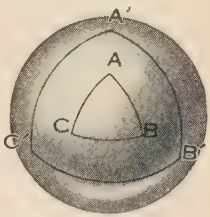
6. The previous figure illustrates the method used in drawing great circles on a sphere. LMH is drawn from A as a pole and if it is a great circle, what is the length of the line AL in terms of OA? Why does a side of the polar triangle of ABC lie in arc LMN?

7. Sketch a spherical triangle and its polar triangle, if the sides of the first are (*a*) 60°, 70°, and 80°; (*b*) 100°, 120°, and 140°. Describe the relative positions of the spherical triangle and its polar triangle for part (*a*) and for part (*b*).

8. Sketch a spherical triangle and its polar triangle, if the sides of the first are (*a*) 60°, 90°, and 90°; (*b*) 70°, 80°, and 90°. Describe the relative positions of these triangles.

9. Sketch a spherical triangle and its polar triangle, if the sides of the first are 40°, 100°, and 120°. How do these triangles differ from those already drawn?

Proposition 8 (C N)



214. Theorem. *If one spherical triangle is the polar triangle of another, the second is the polar triangle of the first.*

Given: $\triangle ABC$ and its polar $\triangle A'B'C'$.

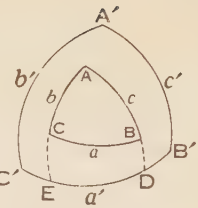
To prove: $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

Proof:

Statements	Facts
1. A is the pole of $\overset{\frown}{B'C'}$ and B is the pole of $\overset{\frown}{A'C'}$.	1. Given.
2. C' is at a quadrant's distance from A and B.	2. § 199.
3. C' is the pole of $\overset{\frown}{AB}$; and in like manner, B' and A' are poles of $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$ respectively.	3. § 201.
4. $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.	4. § 213.

215. Polar Triangles. Two spherical triangles each of which is the polar triangle of the other are called *polar triangles.* In two polar triangles, any arc and the angle at its pole are called *corresponding parts.*

Proposition 9 (C N)



216. Theorem. *The sum of one angle of a spherical triangle and the corresponding side of its polar triangle equals 180°.*

Given: Polar $\triangle ABC$ and $A'B'C'$.

To prove: $\angle A + a' = 180°$, $\angle A' + a = 180°$,
$\angle B + b' = 180°$, $\angle B' + b = 180°$,
$\angle C + c' = 180°$, $\angle C' + c = 180°$.

Proof:

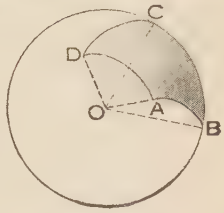
Statements	Facts
1. Produce (if necessary) $\overset{\frown}{AC}$ and $\overset{\frown}{AB}$ to meet $\overset{\frown}{B'C'}$ in E and D respectively.	1. A 32.
2. $\triangle ABC$ and $A'B'C'$ are polar $\triangle$.	2. Why?
3. $\angle A$ = numerically $\overset{\frown}{ED}$.	3. § 210.
4. Any point on arc ABD is at a quadrant's distance from C'; that is $\overset{\frown}{C'E} + \overset{\frown}{ED} = 90°$. In like manner, $\overset{\frown}{ED} + \overset{\frown}{DB'} = 90°$.	4. Why?
5. $\overset{\frown}{ED} + (\overset{\frown}{C'E} + \overset{\frown}{ED} + \overset{\frown}{DB'}) = 180°$, or $\overset{\frown}{ED} + a' = 180°$.	5. Why?
6. $\angle A + a' = 180°$. Similarly $\angle B + b' = 180°$, etc.	6. Why?

Note: While the term supplementary is generally applied to angles, it may be used to state the relation of two arcs or of an angle and an arc whose sum equals 180°. Proposition 9 is sometimes stated: "In two polar triangles, an angle of one is the supplement of the opposite side of the other."

217. Corollary 1. *If two angles of a spherical triangle are equal, the corresponding sides of the polar triangle are equal, and conversely.*

218. Corollary 2. *If one angle of a spherical triangle is equal to an angle of another spherical triangle, the corresponding sides of the respective polar triangles are equal.*

219. Spherical Polygons and Their Polyhedral Angles. The radii from the vertices of a spherical polygon to the center of the sphere form a corresponding polyhedral angle at the center.



The following important relations should be noted:

a) a side of a spherical polygon equals numerically the corresponding face angle of its polyhedral angle, and conversely; (Why?)

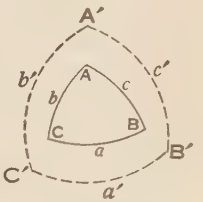
b) an angle of a spherical polygon equals numerically the corresponding dihedral angle of its polyhedral angle, and conversely. (Why?)

220. Theorem A (c n). *Any side of a spherical triangle is less than the sum of the other two sides.* (§§ 84, 219.)

Theorem B (c n). *The perimeter of a convex spherical polygon is less than 360° and greater than 0°.* (§§ 85, 219.)

Proposition 10 (C n)

221. Theorem. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*



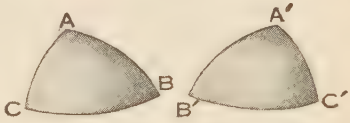
Given: $\triangle ABC$.

To prove: $180° < \angle A + \angle B + \angle C < 540°$.

Proof: STATEMENTS	FACTS
1. Let $\triangle A'B'C'$ be the polar triangle of $\triangle ABC$.	1. § 213.
2. $\angle A + a' = \angle B + b' = \angle C + c' = 180°$.	2. Why?
3. $(\angle A + \angle B + \angle C) + (a' + b' + c') = 540°$.	3. Why?
4. $a' + b' + c' < 360°$ and $> 0°$.	4. § 220*B*.
5. $\angle A + \angle B + \angle C > 180°$ and $< 540°$.	5. Why?

222. Right Spherical Triangles. A spherical triangle, two of whose angles are right angles, is called a *bi-rectangular triangle;* one having three right angles is called a *tri-rectangular triangle.*

223. **Congruent and Symmetric Spherical Triangles.** Spherical triangles (on a sphere or on equal spheres) with the sides and angles of one equal to the respective parts of another and arranged in the same order are called *congruent triangles;* if the parts are arranged in the reverse order, they are called *symmetric triangles.*



Two spherical triangles, whose parts are respectively equal and arranged in the reverse order, cannot be made to coincide because of the curvature of the spherical surfaces. This fact may be illustrated by cutting a pair of triangles (such as ABC and $A'B'C'$) out of the peel of an orange. For this reason, the equality of a pair of symmetric spherical triangles cannot be demonstrated by the Method of Superposition. One exception should be noted. If the two triangles are isosceles spherical triangles, since their parts are respectively equal (whether in the same order or in the reverse order), they can be made to coincide by superposition and are therefore congruent.

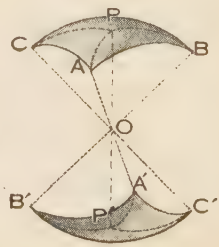
224. **Theorem A.** *Spherical triangles corresponding to a pair of vertical trihedral angles are symmetric.*

Theorem B. *Symmetric spherical triangles may be made to correspond to a pair of vertical trihedral angles.*

Theorem C. *Symmetric isosceles spherical triangles are congruent.*

Proposition 11 (c N)

225. **Theorem.** *Symmetric spherical triangles are equal.*



Given: Symmetric spherical triangles ABC and $A'B'C'$.

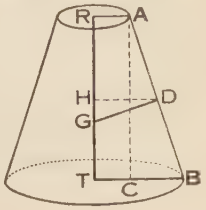
To prove: $\triangle ABC = \triangle A'B'C'$.

Proof: STATEMENTS	FACTS
1. Let $\triangle ABC$ and $\triangle A'B'C'$ be placed so that they correspond to a pair of vertical trihedral angles.	1. § 224*B*.
2. Let POP' be the axis of the small circle through ABC. Let arcs of great circles join P with A, B, and C, and P' with A', B', and C'.	2. §§ 197*e*, 196.
3. $\triangle PAC$ and $\triangle P'A'C'$ are symmetric.	3. § 224*A*.
4. $PA = PC$ and $\angle POA = \angle POC$.	4. § 198, G 16.
5. $\angle POA = \angle P'OA'$, $\angle POC = \angle P'OC'$.	5. Why?
6. $\angle P'OA' = \angle P'OC'$, $P'A' = P'C'$.	6. Why?
7. $\triangle PAC \cong \triangle P'A'C'$; in like manner, $\triangle PAB \cong \triangle P'A'B'$, and $\triangle PBC \cong \triangle P'B'C'$.	7. § 224*C*.
8. $\triangle ABC = \triangle A'B'C'$.	8. § 108.

NOTE: Additional theorems dealing with spherical triangles are found on pp. 127, 128.

Proposition 12

226. Theorem. *The lateral area of a frustum of a cone of revolution equals the product of its axis and the circumference of a circle whose radius is the perpendicular bisector of an element and terminated by the axis.*



Given: A frustum of a cone whose axis is RT. D is the mid-point of slant height AB, $GD \perp AB$, and S is the lateral area of the frustum.

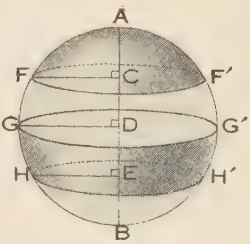
To prove: $S = RT \times 2\pi GD$.

Proof: STATEMENTS	FACTS
1. Let DH be $\perp RT$ and AC be $\perp DH$.	1. Why?
2. $DH = \frac{1}{2}(RA + TB)$ or $2\ DH = (RA + TB)$.	2. Why?
3. $S = \pi \cdot AB \cdot (RA + TB) = AB \cdot 2\ \pi DH$.	3. § 190.
4. $\triangle DHG \sim \triangle ACB$.	4. N 3.
5. $AB : GD = AC : DH$.	5. Why?
6. $AB \cdot DH = AC \cdot GD$.	6. Why?
7. $S = AC \times 2\ \pi GD = RT \times 2\ \pi GD$.	7. Why?

227. Corollary. *The lateral area of a cylinder of revolution or of a cone of revolution equals the product of its axis and the circumference of a circle whose radius is the perpendicular bisector of an element and terminated by the axis.*

NOTE: The statements of §§ 226 and 227 are sometimes combined and given as follows: "The area generated by a line segment revolving about an axis coplanar with it, but not crossing the axis or perpendicular to it, is equal to the projection of the line segment upon the axis multiplied by the circumference of a circle whose radius is the perpendicular bisector of the segment terminating in the axis."

228. Zone. The portion of a sphere included between two parallel planes is called a *zone*. A zone may be regarded as the surface generated by the revolution of an arc of a great circle about its diameter as an axis. The circles which bound the zone are called *bases* and the perpendicular from any point in the plane of one base to the plane of the other base is called an *altitude* of the zone. If one of the planes is tangent to the sphere (cutting it in only one point) the figure formed is called a *zone of one base;* if both planes are tangent to the phere, the zone is the entire sphere. $GG'H'H$ is a zone of two bases with altitude DE; AFF' is a zone of one base, with altitude AC.

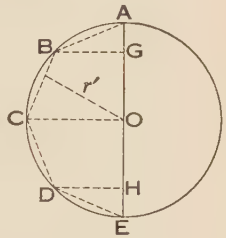


229. Principle. *If one-half a regular polygon of an even number of sides is inscribed in or circumscribed about a semi-*

circle and if the number of sides of the half-polygon is continuously increased without bound and if the complete figure is revolved about the common diameter as an axis, the area (S) of the surface generated by the semicircle and its radius (r) may be substituted in an equation for the corresponding values of the figure generated by either half-polygon.

Proposition 13 (c N)

230. Theorem. *The area of a sphere, whose radius is r, equals $4\pi r^2$.*



Given: A sphere whose radius is r; O is its center, and ACE is a semicircle of a great circle of the sphere.

To prove: Area of the sphere $= 4\pi r^2$.

Proof:

Statements	Facts
1. Let one-half of a regular polygon of an even number of sides be inscribed in the semicircle and let the number of sides which join points in the semicircle be increased without bound, and let the entire figure revolve about AE as an axis. Denote the area of the surface generated by the half-polygon and the apothem in any instance, by S' and r' respectively.	1. O 1.
2. For any such figure, the area of the surface generated is equal to the sum of the lateral areas of cones of revolution, frustums of cones of revolution, and cylinders of revolution, the sum of whose altitudes equals AE, that is, $S' = AE \times 2\pi r'$.	2. §§226, 227, Ax. of Add., Sub.
3. The number of sides of the polygon is increased without bound, $S' \to S$, $r' \to r$, therefore $S = AE \times 2\pi r = 2r \times 2\pi r = 4\pi r^2$.	3. §229, Sub.

NOTE: Statement 2 includes "cylinders of revolution" although the figure used would not require this expression. If half of a regular decagon had been used, however, all three solids of revolution would have been generated.

231. Corollary 1. *The area of a sphere equals four times the area of a great circle of the sphere.*

232. Corollary 2. *The areas of two spheres have the same ratio as the squares of their radii or as the squares of their diameters.*

233. Corollary 3. *The area of a zone equals the product of its altitude and the circumference of a great circle.*

HINT: In § 230, if S is the area of the zone whose altitude is OG (or h), $S = OG \times 2\pi r = 2\pi rh$.

234. Corollary 4. *Zones on the same sphere (or on equal spheres) have the same ratio as their altitudes.*

Sight Exercises. Areas of Spheres

(Find results in terms of π.)

1. Find the area of a sphere whose radius is 1; 2; 3; 4; 5.

2. Find the radius of a sphere whose area is 16π; 64π; 100π.

3. Find the ratio of the areas of two spheres, if their radii are 2 and 3; 2 and 4; 3 and 9.

4. Find the area of a sphere, if the area of a great circle is 9; 12; 25π; x.

5. Find the area of a sphere, if the circumference of a great circle is 6π; 10π; 16π.

6. Find a formula for the area of a sphere in terms of its diameter.

7. Find the ratio of the diameters of two spheres, if their areas are 4π and 9π; 16π and 25π; $3a$ and $12a$.

8. The ratio of the radii of two spheres is 1 : 2. If the area of the smaller sphere is 20π, find the area of the larger. If the area of the larger sphere is 20π, what is the area of the smaller?

9. What is the ratio of the area of the curved surface of a hemisphere to the area of its circular base?

10. Show that the area of a sphere equals the lateral area of the circumscribed cylinder.

11. Show that the area of a sphere equals the area of a circle whose radius is double the radius of a great circle of the sphere.

12. What is the per cent of increase in the area of a sphere, if its diameter is increased (*a*) 20%; (*b*) 50%; (*c*) 100%?

Exercises. Areas of Zones

1. Find the area of a zone of a sphere whose radius is 14 in., if the altitude of the zone is (*a*) 3 in.; (*b*) 7 in.; (*c*) 10 in.; (*d*) 14 in. What special name can be applied to the zone 14 in. high?

2. An iron sphere 7 in. in diameter rests in water which is 4 in. deep. Find the area (*a*) of the dry surface of the sphere; (*b*) of its wet surface. How can the two answers be checked? Apply the check.

3. In Ex. 2, what is the simplest way of finding the ratio of the areas of the zones? What is the ratio?

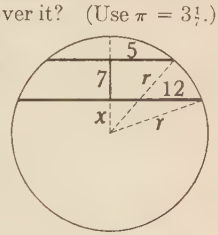
4. What is the ratio of the areas of two zones of the same altitude, if they are on spheres whose radii are respectively 4 in. and 8 in.?

5. If the diameter of a 12-inch sphere is divided into four equal parts by parallel planes, find the areas of the zones thus formed.

6. Find the altitude of a zone, if its area equals the area of a great circle of the sphere.

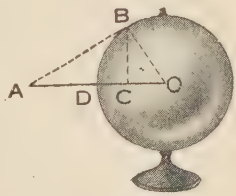
7. The circumference of the base of a hemispherical dome is 66 ft. How many square feet of copper are required to cover it? (Use $\pi = 3\frac{1}{7}$.)

8. The altitude of a zone is 7 in. and the radii of its bases are 5 in. and 12 in. Find the area of the zone, if both bases are on the same side of the center of the sphere.



9. The altitude of a zone is 17 in. and the radii of its bases are 5 in. and 12 in. Find the area of the zone, if the center of the sphere lies between the bases.

10. What part of a globe is visible from a point whose distance from the center equals (*a*) the diameter; (*b*) three times the radius; (*c*) *n* times the radius?



HINT: By what proportion can *CO* be found? How can *CD*, the altitude of the zone be found?

235. Lune. The portion of a sphere bounded by and included between two semicircles of great circles of a sphere is called a *lune*. Either angle formed by the bounding arcs is called the *angle* of the lune.

236. Principle. *Lunes of a sphere (or of equal spheres) have the same ratio as their angles.*

237. Corollary 1. *Lunes of a sphere (or of equal spheres) with equal angles are equal.*

238. Corollary 2. *The ratio of the area of a lune to the area of its sphere equals the ratio of its angle to 360 degrees.*

HINT: The surface of a sphere may be regarded as a lune whose angle is 360°. Let the number of degrees in the angle and the area of the lune be denoted respectively by A and S, then by § 236

$$\frac{S}{4\pi r^2} = \frac{A}{360}.$$

239. Corollary. 3 *If S denotes the area of a lune, A the number of degrees in its angle, and r the radius of the sphere,*

$$S = \frac{A}{90} \times \pi r^2.$$

Exercises. Areas of Lunes

1. Find the area of a lune on a sphere whose radius is 7 in., if the angle of the lune is (*a*) 20°; (*b*) 30°; (*c*) 60°; (*d*) 90°. In each case, the area of the lune is what part of the area of the sphere?

2. The radius of a sphere is 10. Find the angle of the lune whose area is 40π; 50π; 100π.

3. What is the ratio of the areas of two lunes upon the same sphere, if their angles are 35° and 75°?

4. What is the ratio of the areas of two lunes with equal angles, but upon spheres whose radii are 8 and 16?

5. How large is the angle of a lune, if its area is 2 sq. ft. and the area of the sphere is (*a*) 18 sq. ft.; (*b*) 24 sq. ft.; (*c*) 30 sq. ft.?

6. Find the area of the earth's surface bounded by the meridians 60° West Longitude and 80° East Longitude? (Use $r = 4000$ mi.)

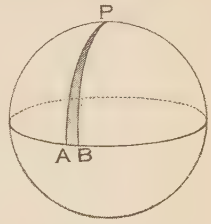
7. If the angle of a lune is 40° and its area is 16π, find the area of a great circle of the sphere.

8. Find the area of a trirectangular spherical triangle of a sphere whose radius is 10.

9. The area of a sphere is 132 sq. in. Find the area of a birectangular spherical triangle whose third angle is 20°.

10. The radius of a sphere is 14 in. Find the angle of a lune whose area equals that of a zone which has an altitude of 4 in.

240. Spherical Degree. That part of a sphere bounded by and included within a spherical triangle two of whose sides are quadrants and whose third side is an arc of 1° is called a *spherical degree.* In the annexed figure, if $PA = PB =$ a quadrant and $AB = 1°$, $\triangle PAB$ represents a spherical degree. *A spherical degree is $\frac{1}{720}$ of a sphere.* Why?



241. Two Ways of Measuring Spherical Surfaces. We learned in plane geometry that an arc of a circle may be measured in the customary units of length and also by arc degrees. For example, we may say that an arc is 22 inches in length or that it is an arc of 60°. In one case, we mean that the arc is composed of twenty-two one-inch arcs and in the second case, we mean that the arc is composed of sixty arcs each of which equals $\frac{1}{360}$ of the circle. In like manner, it is possible to express the area of a spherical surface in two ways. For example, we may say that a spherical triangle contains 616 square inches or that it contains 30 spherical degrees. In the first instance, we mean that the triangle is equal to 616 one-inch squares and in the second, that it is equal to 30 unit triangles each of which is $\frac{1}{720}$ of the sphere. It should be understood that a square inch is a constant standard unit of measure, whereas the spherical degree is a fractional part $(\frac{1}{720})$ of a sphere and its actual size varies with the size of the sphere. The principal value of a spherical degree is that it enables us to relate a spherical polygon to a sphere without our knowing the number of square units in either. For example, if we know that a spherical triangle contains 90 spherical degrees, we know that the triangle is $\frac{90}{720}$ or $\frac{1}{8}$ of the sphere without computing the surface area in square units of either figure.

242. Theorem. *The number of spherical degrees in the area of a lune equals twice the number of degrees in the angle of the lune.*

HINT: A lune whose angle is 1° is how many times a spherical degree (the unit spherical triangle)?

243. Spherical Excess. The sum of the angles of a spherical triangle minus 180° is called the *spherical excess* of the triangle. The spherical excess is usually denoted by E. $[E = \angle A + \angle B + \angle C - 180°.]$

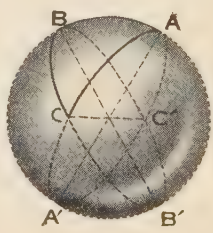
244. Corollary. *The spherical excess of a spherical polygon of n sides equals the sum of its angles minus $(n - 2)\ 180°$.*

Oral Exercises

1. What is a spherical degree?
2. If a sphere contains 2160 sq. in., how many square inches are in a spherical degree?
3. If a spherical triangle contains 90 spherical degrees, by what other name may the figure be called?
4. A hemisphere contains how many spherical degrees? Why?
5. How many spherical degrees in a lune whose angle equals 26° 30′?
6. A trirectangular spherical triangle on a sphere contains how many spherical degrees?
7. What is the spherical excess of a trirectangular spherical triangle?
8. The third angle of a birectangular spherical triangle is 40°. What is its spherical excess? The triangle contains how many spherical degrees?

Proposition 14

245. Theorem. *The area in spherical degrees of a spherical triangle equals its spherical excess.*



Given: $\triangle ABC$ is a spherical triangle whose spherical excess, $\angle A + \angle B + \angle C - 180°$, is denoted by E.

To prove: $\triangle ABC = E$ spherical degrees.

Proof:

STATEMENTS	FACTS
1. Let the great circles whose arcs are sides of $\triangle ABC$ intersect in the diameters, AA', BB', and CC'.	1. § 197*h*.
2. $\triangle ABC + \triangle A'BC$ = lune of $\angle A = 2A$ spherical degrees; $\triangle ABC + \triangle AB'C$ = lune of $\angle B = 2B$ spherical degrees; $\triangle ABC + \triangle ABC'$ = lune of $\angle C = 2C$ spherical degrees.	2. Ax. of Add., § 242.
3. $\triangle ABC' = \triangle A'B'C$.	3. §§ 224*A*, 225.
4. $\triangle ABC + \triangle A'B'C$ = lune of $\angle C = 2C$ spherical degrees.	4. Sub.
5. Adding the first two equations of Statement 2 and Statement 4, $2\triangle ABC + (\triangle ABC + \triangle A'BC + \triangle AB'C + \triangle A'B'C) = 2(A + B + C)$ spherical degrees.	5. Why?
6. $\triangle ABC + \triangle A'BC + \triangle AB'C + \triangle A'B'C$ = a hemisphere = 360 spherical degrees.	6. A 5, § 240.
7. $2\triangle ABC = 2(A + B + C)$ spherical degrees $-$ 360 spherical degrees $= 2(A + B + C - 180)$ spherical degrees.	7. Ax. of Sub.
8. $\triangle ABC = (A + B + C - 180)$ spherical degrees $= E$ spherical degrees.	8. Ax. of Div., Sub.

NOTE: The theorem of § 245 was first used by Cavalieri, the discoverer of the celebrated theorem which bears his name (cf. p. 55) and also by Albert Girard, a famous Dutch mathematician.

246. Corollary 1 (N). *The area in spherical degrees of a spherical polygon equals its spherical excess.*

247. Corollary 2 (c). *A spherical triangle equals a lune whose angle is half the spherical excess of the triangle.*

248. Corollary 2 (c). *A convex spherical polygon equals a lune whose angle is half the spherical excess of the polygon.*

249. Corollary 4. *If S denotes the area of a spherical polygon, E its spherical excess, and r the radius of the sphere,*

$$S = \frac{E}{720} \times 4\pi r^2 = \frac{E}{180} \times \pi r^2.$$

Exercises. Areas of Spherical Polygons

1. What is the spherical excess of a triangle, if the sum of its angles is 240°? What is the area in spherical degrees? Its area is what part of the area of the sphere?

2. What is the spherical excess of a triangle whose angles are (*a*) 80°, 95°, 110°; (*b*) 120°, 135°, 145°; (*c*) 92°, 114°, 126°?

3. If the radius of the sphere is 7 in., what is the area in square inches of each of the triangles in Ex. 2?

4. What is the spherical excess of a spherical hexagon whose angles are 100°, 110°, 120°, 130°, 140°, 150°? Its area is what part of the area of the sphere?

5. The area in spherical degrees of a spherical pentagon is 36. State one set of values for its angles.

6. What is the spherical excess of the pentagon in Ex. 5? Its area is what part of the area of the sphere?

7. The spherical excess of a triangle is 125°. If two of its angles are 105° and 95°, what is the third angle? What is the angle of the lune which is equal to the triangle?

8. If the sides of a spherical triangle are arcs of 80°, 110°, and 130°, what is the spherical excess of its polar triangle? The polar triangle is what part of the sphere?

9. What are the angles of a spherical triangle formed by two meridians on the earth's surface and a 30° arc of the equator? What part of the area of the earth is the triangle?

10. How may a spherical triangle be found upon the earth's surface with two right angles and an angle of 120°? Such a triangle is what part of the earth's surface?

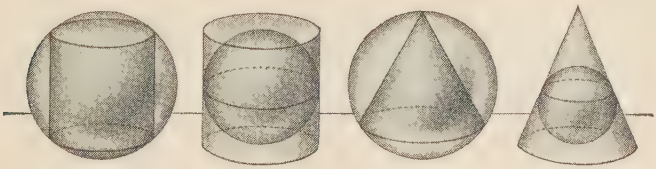
11. A spherical triangle, whose angles are in the ratio of 1 : 2 : 3, is one-tenth the area of a sphere. How many degrees are in each angle of the triangle?

12. In $S = \frac{E}{180} \times \pi r^2$, solve for r. By using this formula, make up a problem which will have $r = 7$ for its answer.

13. Make up a problem of your own which uses the Fact of § 247 in its solution.

250. Inscriptible and Circumscriptible Solids. In §§ 206 and 207, it was proved that spheres can be inscribed in and circumscribed about any tetrahedron. It is possible to prove that a sphere can be inscribed in and circumscribed about any one of the five regular polyhedrons. To do this, however, would involve the drawing of extremely complicated figures and proofs which are beyond the scope of this text. The proofs for the tetrahedron (§§206, 207) were illustrative of what is required. It is also possible to prove that any of the five regular polyhedrons can be inscribed in and circumscribed about a given sphere, but for the reasons just stated the proofs are omitted. The following assumption may, therefore, be made: *Any of the five regular polyhedrons and a sphere are mutually inscriptible and circumscriptible.*

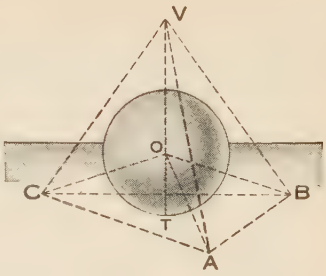
It will be assumed in a number of exercises which are to follow that there are certain cylinders and cones which can be inscribed in and circumscribed about spheres. Such solids are not to be regarded as general figures of their kind for they possess special properties which are illustrated by the following figures:



251. Principle. *If a regular polyhedron is inscribed in or circumscribed about a sphere and if the total number of faces of the polyhedron is continuously increased without bound, the area (S) and the volume (V) of the sphere may be substituted in an equation for the corresponding values of either polyhedron.*

Proposition 15 (c N)

252. Theorem. *The volume of a sphere equals one-third the product of its radius and its area.*



Given: A sphere with radius OT denoted by r, its area by S, and its volume by V.

To prove: $V = \frac{1}{3}rS$.

Proof:

Statements	Facts
1. Let a regular tetrahedron be circumscribed about the sphere and let the total number of faces increase without bound. Denote the total area and the volume of the polyhedron, in any instance, by S' and V'.	1. § 250.
2. Connect O with the vertices of any such polyhedron. The sum of the volumes of the pyramids thus formed equals $\frac{1}{3} \times r \times$ the sum of the areas of all the faces or $V' = \frac{1}{3}rS'$.	2. Why?
3. The total number of faces increases without bound, $S' \to S$, $V' \to V$, therefore $V = \frac{1}{3}rS$.	3. § 251.

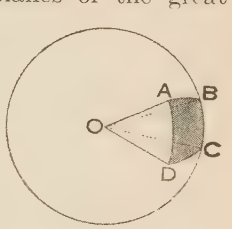
Note: If planes are passed perpendicular to OC, OV, OB, and OA at the points where these lines intersect the surface of the sphere, a polyhedron of eight faces is formed. If this process is continuously repeated, polyhedrons of an increasing number of faces are formed. It should be understood that the regular tetrahedron is the only regular polyhedron of the circumscribed polyhedrons in the development of § 252. The polyhedron of eight faces, for example, is not a regular octahedron, because four of its faces are triangles and the other four are hexagons.

253. Corollary 1 (c). *If V denotes the volume and r the radius of a sphere,* $V = \frac{4}{3}\pi r^3$.

254. Corollary 2. *The volumes of two spheres have the same ratio as the cubes of their radii or as the cubes of their diameters.*

$$\frac{V}{V_1} = \frac{r^3}{r_1^3} = \frac{d^3}{d_1^3}.$$

255. Spherical Pyramid. The figure formed by a spherical polygon and the intersections of the planes of the great circles forming the sides of the polygon is called a *spherical pyramid;* the spherical polygon is called the *base,* the radius of the sphere is called the *altitude,* and the center of the sphere is called the *vertex* of the pyramid.



256. Theorem. *If the volume of a spherical pyramid is denoted by V and the area of the base by B, $V = \frac{1}{3} Br$.*

HINT: The volume of a sphere may be regarded as the sum of the volumes of a number of spherical pyramids. The volume of any spherical pyramid and the volume of the sphere have the same ratio as the area of the base of the pyramid and the area of the sphere.

$$V : \tfrac{4}{3}\pi r^3 = B : 4\pi r^2 \text{ or } V = \tfrac{1}{3} Br.$$

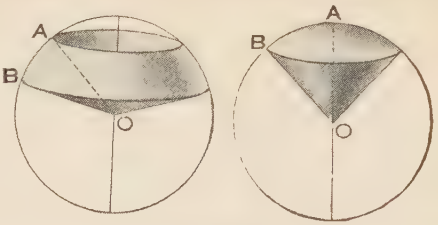
257. Spherical Wedge. The figure formed by a lune and the intersection of the planes of the great circles forming the sides of the lune is called a *spherical wedge.* The bounding lune is called the *base* of the spherical wedge.

258. Theorem. *If the volume of a spherical wedge is denoted by V and the number of degrees in the angle of its bounding lune by A, $V = \frac{A\,\pi r^3}{270}$.*

HINT: $V : \frac{4}{3}\pi r^3 = A : 360$ or $V = \frac{A\,\pi r^3}{270}$.

259. Spherical Sector and Spherical Cone. The closed figure generated by a sector of a circle which revolves about a diameter of the circle as an axis is called a *spherical sector;*

if a radius of the plane sector is a part of the axis, the figure is called a *spherical cone*. The zone generated by the arc of the circular sector is called the *base* for either figure.



260. Theorem. *If the volume of a spherical sector is denoted by V, the altitude of its base by h, and the radius by r, $V = \frac{2}{3}\pi r^2 h$.*

HINT: $V : \frac{4}{3}\pi r^3 = 2\pi rh : 4\pi r^2$ or $V = \frac{2}{3}\pi r^2 h$.

261. Spherical Segment. The closed figure formed by a zone and the portions of the planes included within the bases of the zone is called a *spherical segment*. If the zone has only one base, the figure is called a *spherical segment of one base*. The altitude and bases of the zone are called respectively the *altitude* and *bases* of the spherical segment.

262. Theorem. *If the volume of a spherical segment is denoted by V, the radii of the bases by r and r_1, and the altitude by h, $V = \frac{1}{6}\pi h(3r^2 + 3r_1^2 + h^2)$.*

HINT: The derivation of this formula is not required by many syllabi and may be given whatever emphasis seems desirable. An interesting proof may be had by applying the Prismatoid Formula.

Exercises. Volumes of Spheres

1. Find the volume of a sphere in terms of π, if $r = 2;\ 3;\ 5;\ \frac{a}{2};\ \sqrt{a}$.

2. Find the volume of a sphere whose area is $36\pi;\ 100\pi;\ 144\pi;\ 8\pi;\ \pi$.

3. Find the radius of a sphere whose volume is numerically equal to its area.

4. Find the radius of a sphere whose volume is $2304\pi;\ 4500\pi$.

5. Find the ratio of the volumes of two spheres whose radii are 6 and 3; 6 and 2; 6 and 1.

6. How many two-inch spheres are equal to an eight-inch sphere? Compare the total area of the smaller spheres with the larger one.

7. If the radius of a sphere is doubled, how are the area and volume of the sphere changed?

8. If the radius of a four-inch sphere is increased by one inch, what is the per cent of increase in volume? in area?

9. A solid iron ball 4 in. in diameter is worth 80 cents. At the same rate, how much is a ball 8 in. in diameter worth?

10. What is the ratio of the diameters of two spheres, if the ratio of their volumes is 1 : 8? 8 : 27? $a^3 : b^3$?

11. Find the ratio of the areas of two spheres, if the volumes are $2304\,\pi$ and $4500\,\pi$.

12. The weight of a cubic inch of cast iron is about 4 oz. Find the weight of a solid iron ball which is 6 in. in diameter.

13. The volumes of two spheres have the ratio 8 : 27. If the volume of the smaller is 216, find the volume of the larger. If the volume of the larger is 216, what is the volume of the smaller?

14. A hollow iron ball whose outside and inside diameters are 12 in. and 6 in. respectively is melted and molded into a cylinder 7 in. high. What is the diameter of the cylinder?

15. The radius of the base of a right circular cone and the radius of a sphere are equal. What is the height of the cone, if the sphere and the cone have the same volume?

Exercises. Spherical Pyramids, Sectors, Wedges, and Segments

1. What fractional part of the volume of a sphere is a spherical pyramid whose base has a spherical excess of 216°?

2. Find the volume of a spherical pyramid, if the radius of its sphere is 12 in. and the base of the pyramid is a spherical triangle whose angles are 82°, 96°, and 118°.

3. Find the volume of a spherical pyramid whose altitude is 6 and the angles of whose base are 80°, 94°, 102°, and 116°.

4. The volume of a spherical pyramid is $108\,\pi$ and the spherical excess of its base is 270°. What is the radius of the sphere?

5. Find the volume of a spherical sector, if its altitude is 8 and the radius of the sphere is 10.

6. What is the volume of a spherical sector whose altitude is a radius of the sphere? What is another name for such a spherical sector? Does the formula for the volume of this figure also apply?

7. Find the volume of a spherical sector of a 12-inch sphere whose base is a zone formed by parallel planes 3 in. and 4 in. respectively from the center of the sphere and lying on the same side of it.

8. Find the volume of the spherical sector which would be formed in Ex. 7, if the planes were parallel and on oposite sides of the center.

9. Find the volume of a spherical wedge of a 10-inch sphere, if the angle of the wedge is 40°.

10. Find the volume of a spherical wedge, if the perimeter of its base is 6π and the angle of the wedge is 120°.

11. Using the formula of § 262, find the volume of a spherical segment of one base whose radius is equal to the altitude of the segment. What is another name for such a figure?

12. Three parallel planes each perpendicular to a diameter of a sphere divide the diameter into four equal parts. Find the volume (in terms of r and π) of each of the spherical segments thus formed.

Exercises. Inscribed and Circumscribed Spheres

1. The edge of a cube is 14 in. Find the area of the inscribed sphere. What is the volume of the sphere?

2. The edge of a cube is 14 in. Find the area of the circumscribed sphere. What is the volume of the sphere?

3. Spheres are inscribed in and circumscribed about a cube whose edge is a. Find the ratio of the diameters of the spheres; of the areas; of the volumes.

4. Cubes are inscribed in and circumscribed about a sphere whose radius is a. Find the ratio of the edges of the cubes; of their total areas; of their volumes.

5. A sphere is circumscribed about a right prism 8 in. high with a square base 6 in. by 6 in. Find the area and volume of the sphere.

6. The altitude of a right circular cone is 16 in. and the radius of the inscribed sphere is 6 in. What is the radius of the base of the cone?

7. Show that the volume of a tetrahedron equals one-third the product of its total area and the radius of the inscribed sphere. What is the corresponding theorem in plane geometry?

8. An inverted cone is 4 in. across the top and 10 in. deep. How far into the cone will a ball drop if its diameter is 2 in.? if its diameter is 3 in.?

9. If a 12-inch ball fits snugly in a 12-inch cubical box, how large a ball could still be put into each corner between the large ball and the box?

10. If spheres are inscribed in and circumscribed about a regular tetrahedron, find the ratio of the diameters of the spheres; of their areas; of their volumes?

11. If regular tetrahedrons are inscribed in and circumscribed about a sphere, find the ratio of the edges of the tetrahedrons; of their areas; of their volumes.

Additional Theorems. Spherical Triangles

NOTE: The theorems grouped here are subsidiary or auxiliary theorems of various syllabi. Some teachers may care to omit their formal proofs while others may wish to give these theorems as original exercises.

A (c n). *Spherical triangles on a sphere (or on equal spheres) (a) with two sides and the included angle of one equal to the respective parts of another; (b) with two angles and the included side of one equal to the respective parts of another, are congruent or symmetric, according as their parts are arranged in the same or reverse order.*

HINT: Prove the corresponding trihedral angles congruent, or use the Method of Superposition.

B (c). *Base angles of an isosceles spherical triangle are equal, and conversely.*

HINT: Can the direct theorem be proved by the same method as is used in plane geometry? How would you apply § 217 to prove the converse?

C (c n). *Spherical triangles on a sphere (or on equal spheres) which are mutually equilateral are congruent or symmetric, according as their parts are arranged in the same or reverse order.*

HINT: Can this theorem be proved by means of the corresponding trihedral angles? Can it be proved by the same method as is used for the corresponding theorem of plane geometry?

D (c n). *Spherical triangles on a sphere (or on equal spheres) which are mutually equiangular are congruent or symmetric, according as their parts are arranged in the same or reverse order.*

HINT: Prove that the respective polar triangles are mutually equilateral and apply Theorem C.

E (c). *If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal and the side opposite the greater angle is the greater.*

HINT: Use the same method as with the corresponding theorem of plane geometry.

F (c). *If two sides of a spherical triangle are unequal, the angles opposite these sides are unequal and the angle opposite the greater side is the greater.*

HINT: Use the Exclusion Method of Proof.

Miscellaneous Propositions

1. Sections of a sphere (or of equal spheres) which are unequally distant from the center are unequal and the more remote from the center is the smaller, and conversely.

2. A great circle drawn to a second great circle from the latter's pole is perpendicular to the second great circle.

3. If a plane is tangent to a sphere, a perpendicular to the plane at the point of contact passes through the center of the sphere.

4. A straight line tangent to a circle of a sphere lies in the plane tangent to the sphere at the point of contact.

5. A straight line perpendicular to the radius of a sphere at its outer extremity is tangent to the sphere.

6. Lines which are tangent to a sphere from a given external point are equal and the points of tangency lie in a circle.

7. The intersection of two spheres is a circle whose center lies in the straight line joining the centers of the spheres and whose plane is perpendicular to that line.

8. A sphere is determined by four non-coplanar points, no three of which are collinear.

9. A spherical angle, the plane angle of the tangents to the arcs of the angle at its vertex, the dihedral angle of the planes of the arcs, and the great circle arc having the vertex of the angle as a pole and included between its sides are all numerically equal.

10. A spherical triangle can have one, two, or three right angles, or one, two, or three obtuse angles.

11. The perimeter of a convex spherical polygon is less than a great circle of the sphere.

12. Two spherical triangles whose vertices are the opposite ends of diameters are symmetric and equal.

13. If the volume of a spherical pyramid is denoted by V and the spherical excess of its base by E, $V = \frac{\pi r^3 E}{540}$.

14. If the volume of a spherical segment of one base is denoted by V, the altitude by h, and the radius of the base by r, $V = \frac{1}{6}\pi h(3r^2 + h^2)$.

Review Oral Exercises

1. How many spheres may pass through one point? two points? three points? four points? Discuss the limitations in any case.

2. What figure is formed by the intersection of a plane and a sphere? of two spheres?

3. Through a point on a sphere, how many tangent lines may be drawn? how many tangent planes? May two of the tangent lines determine one tangent plane?

4. Through two points on a sphere, how many small circles may be drawn? Explain why there is only one great circle through the two points?

5. Through an external point to a sphere how many tangent lines are there? how many tangent planes are there?

6. Under what condition will a great circle and a small circle of a sphere have the same poles? Is each of these circles at a quadrant's distance from the poles?

7. Can a sphere be inscribed in and circumscribed about any polyhedron? For what polyhedrons is this possible?

8. Can a sphere be inscribed in and circumscribed about a cylinder? Can you name any other solids for which this is true?

9. What are the limitations of the sum of the sides of a convex spherical polygon?

10. What are the limitations of the sum of the angles of a spherical triangle?

11. How is a spherical angle measured? Explain.

12. What are polar triangles?

13. State two theorems dealing with two polar triangles.

14. What relations exist between a spherical polygon and its polyhedral angle at the center of the sphere?

15. What are birectangular and trirectangular spherical triangles?

16. Is it possible to have a spherical parallelogram? a spherical trapezoid? a spherical square? Why?

17. What is the difference between congruent and symmetric spherical triangles?

18. Are symmetric isosceles spherical triangles congruent? Explain.

19. Are spherical triangles and spherical quadrilaterals which are equilateral also equiangular? Are they mutually equiangular?

20. What is a spherical degree? A hemisphere has how many spherical degrees? What is the spherical excess of a polygon?

21. What is a zone? a lune? May a hemisphere be called a zone? a lune?

22. What zones on the earth's surface have only one base? Name the zones of two bases.

23. What part of the earth's surface is included between the 20th and the 30th meridians? Are there two answers? Why? Can either be found in more than one way?

24. Can the area of a spherical triangle be found, if its angles are known? if its sides are known?

25. State the formula for the area of a zone of two bases; of one base.

26. What areas are denoted by each of the following: (*a*) $2\ \pi rh$; (*b*) $4\ \pi r^2$; (*c*) $\frac{A}{90}\pi r^2$; (*d*) $\frac{\pi r^2 E}{180}$?

27. What volumes are denoted be each of the following: (*a*) $\frac{1}{3}Sr$; (*b*) $\frac{4}{3}\pi r^3$; (*c*) $\frac{2}{3}\pi r^2 h$?

28. What factors do you need to know to find the area (*a*) of a spherical triangle; (*b*) of a sphere; (*c*) of a lune; (*d*) of a zone?

29. What factors do you need to know to find the volume (*a*) of a sphere; (*b*) of a spherical pyramid; (*c*) of a spherical sector; (*d*) of a spherical cone; (*e*) of a spherical wedge; (*f*) of a spherical segment?

30. If the radii of two spheres are 3 and 2 respectively, what is the ratio (*a*) of the circumferences of their great circles; (*b*) of the areas of the spheres; (*c*) of the volumes?

31. If the radius of a sphere is 5 in., what is the radius of a small circle which is (*a*) 3 in. from the center; (*b*) 4 in. from the center?

32. Are all meridians on the earth's surface perpendicular to the equator? Why?

33. Find the area and volume of a sphere whose radius is three inches.

34. The ratio of the areas of two spheres is 18 : 50. The smaller sphere has a 3-inch radius. What is the volume of the larger sphere?

35. Does the "Postulate of Parallels" of plane geometry hold true for spherical geometry? Why?

36. If the radius of a sphere is doubled, how is the area changed? How does the volume change?

37. If the diameter of the earth were increased by one mile, by how much would the equator increase?

38. How does the area of a sphere whose radius is r compare with the area of a circle of radius $2r$?

39. How would you find the total area of a spherical wedge?

40. State five theorems of Chapter Four which are required as "standard book propositions" by the College Entrance Examination Board.

Miscellaneous Exercises

1. What is the locus of a line tangent to a sphere from an external point? Find the area of the surface of this locus, if the line is 12 in. long and the radius of the sphere is 5 in.

2. Find the radius of a sphere equal in area to two spheres whose radii are 6 in. and 8 in.? What is the volume of the largest sphere?

3. The area of a lune is ten spherical degrees. If the sphere has a radius of 10 in., find the angle of the lune and its area in square inches.

4. If a right circular cylinder is circumscribed about a sphere, their total areas have the same ratio as their volumes.

5. In plane geometry, we prove that "base angles of an isosceles triangle are equal." State and prove the corresponding theorem for spherical geometry.

6. The radius and altitude of a cone of revolution are 8 and 15 respectively. Find the area and volume of the inscribed sphere.

7. Find the volume of a spherical pyramid whose base is a trirectangular triangle and whose altitude is r.

8. Two sides of a parallelogram are 8 and 16 and include an angle of 150°. Find the volume and total area of the figure which is generated by revolving the parallelogram about the side 16 as an axis.

9. If the parallelogram of Ex. 8 is revolved about the side 8 as an axis, find the total area and volume of the figure generated.

10. Find the total area and volume of the figure generated by revolving a trapezoid, whose upper base is 10 and whose lower base angles are 45° and 30°, about its lower base as an axis.

11. Find the total area and volume of the figure generated by revolving the trapezoid of Ex. 10 about its upper base as an axis.

12. The radius of a sphere is r and the plane area bounded by a small circle of the sphere is S. How far is the plane of the small circle from the center of the sphere?

13. Find the area of a zone of one base generated by an arc of 45°, if the radius of the sphere is 10.

14. Two concentric spheres whose radii are 10 in. and 8 in. respectively, have a trihedral angle at the common center. If the three face angles

of the trihedral angle are 45°, 60°, and 90° respectively, find the lengths of the sides of the two corresponding spherical triangles, one on each sphere.

15. A right triangle whose legs are 3 in. and 4 in. has a line parallel to the longer leg and 6 in. distant from it. Find the total area and volume of the figure generated by revolving the right triangle about the line as an axis.

16. The line of centers of two circles is 25 in. long and the radii of the circles are 15 in. and 20 in. respectively. If the complete figure is revolved about the line of centers produced as an axis, find the total area and volume of the figure generated.

17. Find the area of a spherical triangle on a sphere whose radius is 6 in., if the perimeter of the polar triangle is 180°.

19. The sum of the angles of a spherical hexagon is greater than 720° and less than 2680°. Prove.

20. The line of centers of two circles is 25 in. long and the radii are 5 in. and 12 in. respectively. If the common external tangents are drawn and the complete figure is revolved about the line of centers produced as an axis, find the volume of the figure generated.

21. Find the area of the external surface generated in Ex. 20.

A SPHERICAL BALLOON

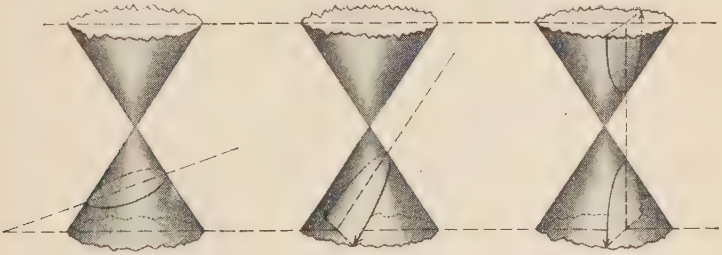
The most popular balloons of this type are the 80,000 cu. ft.–53 ft. diameter, and the 35,000 cu. ft.–42 ft. diameter, the former being used in the International Race and the latter in the National Race. How do the actual spherical volumes compare with the given approximate volumes?

Courtesy Goodyear-Zeppelin Corp.

SUPPLEMENT

CONIC SECTIONS

The curves formed by planes which cut a cone are called conic sections. The names of these sections are the ellipse, the parabola, and the hyperbola. The annexed figures illustrate the manner by which they are formed.



Ellipse Parabola Hyperbola

If a cutting plane intersects each element of a cone, an ellipse is formed (of which the circle is a special ellipse); if the plane is parallel to one and only one element, a parabola is formed; if the plane cuts both nappes of the cone, a curve of two distinct branches called a hyperbola is formed. Under certain conditions, these curves may degenerate into a point, one straight line, two intersecting lines, or two parallel lines.

Menaechmus (375–325 B.C.), who was the geometry tutor of Alexander the Great, discovered the conic sections by cutting a given fixed plane by three different cones. He found that if the fixed plane were cut by an acute-angled cone, a right-angled cone, and an obtuse-angled cone, the curves which we call respectively the ellipse, the parabola, and the hyperbola were formed. These curves for a long while were called "the Menaechmian Triads." Later Apollonius (260–200 B.C.) so thoroughly investigated the properties of these curves that very little was left for later mathematicians to prove by purely geometric methods. However, due to the discoveries and the theories developed by Kepler, Desargues, Descartes, Fermat, and others, many new properties of these curves have been found by the analytic, the descriptive, and the projective geometries.

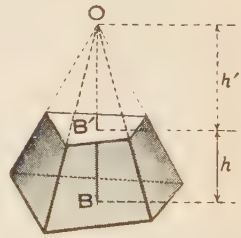
A list of a few of the properties and applications of conic sections follow. The ellipse is the locus of points the sum of whose distances from two fixed points is a constant. The parabola is the locus of points which are equally distant from a fixed point and a fixed line. The hyperbola is the locus of points the difference of whose distances from two fixed points is a constant. Planets move around the sun in orbits which are ellipses. The theoretical path of a projectile is a parabola. The designs of many arches show ellipses or parabolas. Some problems of mechanics are best solved by applying known properties of the hyperbola. The reflectors of searchlights are constructed according to an important property of parabolas.

Supplementary Proof

Theorem. *The volume of a frustum of a pyramid equals one-third the product of the altitude and the sum of the bases and their mean proportional.* [147*a*]

Given: $\boldsymbol{F}$ is a frustum of a pyramid with volume V, altitude h, bases B and B'.

To prove: $V = \frac{1}{3}\, h(B + B' + \sqrt{BB'})$.

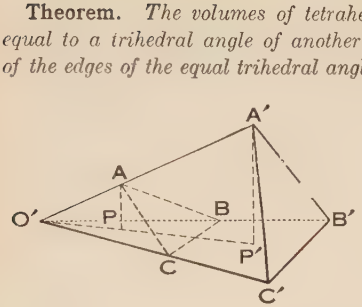


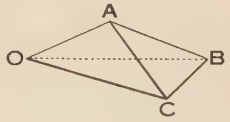
Proof:

STATEMENTS	FACTS
Let O-B denote the pyramid of which $\boldsymbol{F}$ is the frustum. Let $h + h'$ represent the altitude of the pyramid.	
1. Volume of $\boldsymbol{F}$ = volume of O-B − volume of O-B'.	1. A 5, 7.
2. Volume of O-$B = \frac{1}{3}(h + h')B$; volume of O-$B' = \frac{1}{3}\, h'B$.	2. § 142.
3. Volume of $\boldsymbol{F} = \frac{1}{3}(h + h')B - \frac{1}{3}\, h'B'$ or $V = \frac{1}{3}\, hB + \frac{1}{3}\, h'(B - B')$.	3. Sub.
4. $\dfrac{B}{B'} = \dfrac{(h + h')^2}{h'^2}$.	4. § 133.
5. $\dfrac{\sqrt{B}}{\sqrt{B'}} = \dfrac{h + h'}{h'}$.	5. A 10.
6. $\dfrac{\sqrt{B} - \sqrt{B'}}{\sqrt{B'}} = \dfrac{h}{h'}$.	6. L 1.
7. $h'(\sqrt{B} - \sqrt{B'}) = h\sqrt{B'}$.	7. L 2.
8. $\sqrt{B} + \sqrt{B'} = \sqrt{B} + \sqrt{B'}$.	8. Iden.
9. $h'(B - B') = h(\sqrt{BB'} + B')$.	9. A 8.
10. $V = \frac{1}{3}\, hB + \frac{1}{3}\, h(\sqrt{BB'} + B') = \frac{1}{3}\, h(B + B' + \sqrt{BB'})$.	10. Sub.

Supplementary Proof

Theorem. *The volumes of tetrahedrons with a trihedral angle of one equal to a trihedral angle of another have the same ratio as the products of the edges of the equal trihedral angles.* [147*b*]





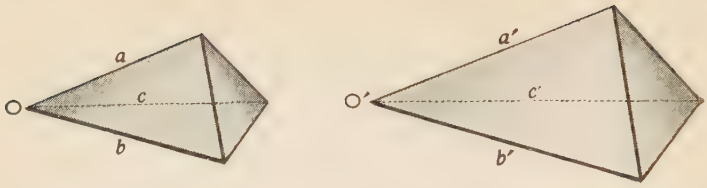
Given: V and V' are the volumes of tetrahedrons $O\text{-}ABC$ and $O'\text{-}A'B'C'$ with equal trihedral angles O and O'.

To prove: $\dfrac{V}{V'} = \dfrac{OA \cdot OB \cdot OC}{O'A' \cdot O'B' \cdot O'C'} =$

Proof: STATEMENTS	FACTS
1. Place the tetrahedrons so that the equal trihedral angles O and O' coincide.	1. Given, A 33.
2. Draw altitudes AP and $A'P'$.	2. § 19.
3. $\dfrac{V}{V'} = \dfrac{\triangle O'BC \cdot AP}{\triangle O'B'C' \cdot A'P'} = \dfrac{\triangle O'BC}{\triangle O'B'C'} \times \dfrac{AP}{A'P'}$.	3. § 143*a*.
4. $\dfrac{\triangle O'BC}{\triangle O'B'C'} = \dfrac{O'B \cdot O'C}{O'B' \cdot O'C'}$.	4. M 5.
5. $\perp$s AP and $A'P'$ are $\parallel$.	5. § 25.
6. $\parallel$s AP and $A'P'$ determine a plane intersecting plane $O'B'C'$ in the straight line $O'PP'$.	6. §§ 5*d*, 8.
7. $\triangle O'AP \sim \triangle O'A'P'$.	7. N 9.
8. $\dfrac{AP}{A'P'} = \dfrac{O'A}{O'A'} = \dfrac{OA}{O'A'}$.	8. M 10.
9. $\dfrac{V}{V'} = \dfrac{O'B \cdot O'C}{O'B' \cdot O'C'} \times \dfrac{O'A}{O'A'} = \dfrac{OB \cdot OC \cdot OA}{O'B' \cdot O'C' \cdot O'A'}$.	9. Sub.

Supplementary Proof

Theorem. *The volumes of similar tetrahedrons have the same ratio as the cubes of any pair of corresponding edges.* [147*c*]



Given: V and V' are the volumes of similar tetrahedrons O and O' with a and a' two corresponding edges.

To prove: $\frac{V}{V'} = \frac{a^3}{a'^3}$.

Proof:

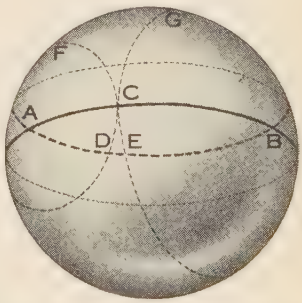
Statements	Facts
1. Tetrahedron $O \sim$ tetrahedron O'.	1. Given.
2. Trihedral $\angle O$ = trihedral $\angle O'$.	2. § 146.
3. $\frac{V}{V'} = \frac{abc}{a'b'c'} = \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'}$.	3. § 147*b*.
4. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.	4. M 10, Sub.
5. $\frac{V}{V'} = \frac{a}{a'} \cdot \frac{a}{a'} \cdot \frac{a}{a'} = \frac{a^3}{a'^3}$.	5. Sub.

Supplementary Proof

Theorem. *The shortest line on a sphere between two points is the minor arc of a great circle which passes through those points.*

Given: A and B are two points on a sphere and ACB is the minor arc of a great circle through A and B.

To prove: ACB is the shortest line on the sphere between A and B.



Proof: Let C be *any point* on ACB and let FCD and GCE be two small circles whose poles are respectively A and B and whose polar distances are measured by AC and BC respectively. These two small circles cannot intersect in any other point than C, for if they should meet in some other point X, then in the triangle AXB which would be formed, $AX + XB$ would not be greater than ACB. Therefore, any line from A to B which does not pass through C must intersect the

small circles in distinct points. Let $ADEB$ be such a line and let D and E be the respective points of intersection. Whatever the nature of the line AD, there is a line of equal length from A to C; and in like manner, a line equal to BE from B to C. That is, there is a line from A to B through C which is less than line $ADEB$ by the length of DE. Since, any line ($ADEB$) which does not pass through C cannot be the shortest line from A to B and since C is *any point on* ACB, the shortest line passes through *every point on* ACB. That is, ACB is the shortest line between A and B.

GENERAL MISCELLANEOUS EXERCISES

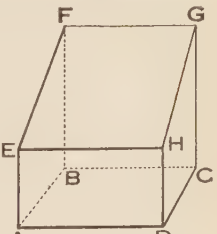
1. The base of a right parallelopiped is a rhombus of side 12.3″ and acute angle 55°. The length of the lateral edge is 16.8″. Find the area of the total surface.

2. Compute the length of the longest diagonal of the right parallelopiped of Ex. 1.

3. Calculate the volume of the figure of Ex. 1.

4. Find the volume and the total area of a regular pyramid with the lateral edge equal to 20″ and each side of the base equal to 10″, if the base is (*a*) triangular; (*b*) square; (*c*) pentagonal; (*d*) hexagonal; (*e*) octagonal.

5. The figure represents a truncated prism with a square base. The lateral edges are perpendicular to the base. $AB = 10$, $AE = 6$, $BF = 10$, $CG = 10$, $DH = 6$. Compute the value of the following: (*a*) the length of the diagonal FD; (*b*) the number of degrees in the face angle EFB; (*c*) the area of each face; (*d*) the number of degrees in the angle BFH; (*e*) the volume.

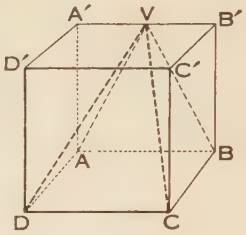


6. The base of a pyramid is a rectangle $ABCD$. The lateral edge VA is perpendicular to the base. $AB = 6$, $BC = 8$, $VA = 10$. Find the following: (*a*) the number of degrees in any face angle; (*b*) the number of degrees in the angle which BV (or CV or DV) makes with the plane $ABCD$; (*c*) the number of degrees in the dihedral angles formed by $ABCD$ with each of the other four planes; (*d*) the area of the total surface.

7. If the pyramid of Ex. 6 is made of wood and weighs 5 pounds, how far from the vertex must it be cut by a plane parallel to the base so that the frustum cut off will weigh 3 pounds?

8. A frustum of the pyramid of Ex. 6 is formed by a plane 6″ from the base. Find (*a*) the number of degrees in the face angles; (*b*) the total area; (*c*) the volume.

9. The accompanying figure is a cube of edge 12 in. and V is the mid-point of $A'B'$. (*a*) Find the lengths of VA and VD. (*b*) Compute the number of degrees in the angles of the triangles VAB, VBC, and VCD. (*c*) What angle in the figure is a plane angle of the dihedral angle formed by $ABCD$ with VBC? (*d*) Explain a method for constructing a plane angle of the dihedral angle formed by $ABCD$ with VCD (*e*) How many degrees are in such an angle? (*f*) Compute the area of each face of the pyramid.



10. Each side of the lower base of the frustum of a regular pyramid is 7, each side of the upper base is 3, and the altitude is 4. Find the volume and the total area of the frustum if its bases are (*a*) triangular; (*b*) square; (*c*) pentagonal; (*d*) hexagonal; (*e*) octagonal.

11. A vat, open at the top, is to be built in the form of a frustum of a cone to contain 1000 cu. ft. The diameter of the upper base is to be 12′ and of the lower base 8′. Calculate the number of square feet of galvanized iron required, assuming that 15 per cent of the iron purchased will be wasted or used for joints.

12. A roof is in the form of a frustum of a cone, the slope of which is an angle of 45° with the horizontal. The diameter at the bottom is 31.5 ft. and at the top is 15.5 ft. Calculate the number of square feet in the roof.

13. Two lead balls, each one inch in diameter, are melted and cast in a single sphere. What is the diameter of the latter?

14. If a spherical drop of water is vaporized into a spray consisting of a thousand equal spheres without altering the volume, by what per cent has the total surface been increased?

15. The area of the entire surface of a solid polyhedron weighing 64 lb. is 340 sq. in. What is the surface of a similar polyhedron made of the same material and weighing 1000 lb.?

16. It is desired to double the capacity of a spherical balloon. By what per cent must the area of canvas be increased?

17. Find the volume of the largest sphere which can be cut from a metal cone, the diameter of whose base is 6″ and whose slant height is 6″.

18. Find the total area and the volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 6″ and whose altitude is 8″.

19. The diameter of a sphere is 18. Calculate the volume of the largest regular prism of altitude 15 which can be cut from the sphere if the base of the prism is (*a*) triangular; (*b*) square; (*c*) pentagonal; (*d*) hexagonal; (*e*) octagonal.

20. Describe the locus of points whose distances from a fixed plane and a fixed line perpendicular to the plane are in a given ratio.

21. P is a horizontal plane, Q is a vertical plane, and A is a point in the line of intersection of P and Q. x is a line in Q perpendicular to P at A and y is a line in P perpendicular to Q at A. Describe the locus of points in space which are (*a*) 5″ from A and 3″ from P; (*b*) 5″ from A and 3″ from x; (*c*) equidistant from P and Q; (*d*) equidistant from x and y; (*e*) 5″ from A and equidistant from P and Q; (*f*) 5″ from A and 3″ from Q.

22. In a fixed plane M a line x is drawn and a point A is marked on x. Describe the locus of the points P in space (*a*) such that the perpendiculars from P on x always make an angle of 25° with M; (*b*) such that AP always makes an angle of 18° with X.

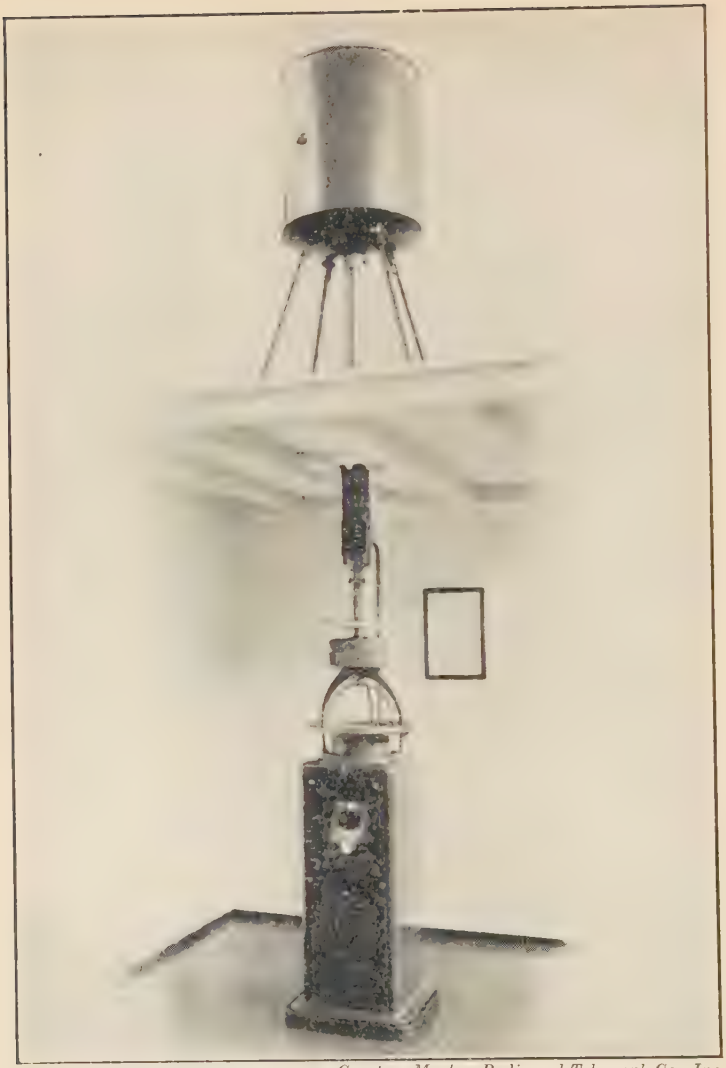
NOTE: The General Miscellaneous Exercises are adapted from the problems in the Appendix of a syllabus of the College Entrance Examination Board. As stated in the syllabus, **the problems are intended to show the kind of training expected of a candidate in solid geometry and should not be looked upon as typical examination questions.** It may be advisable for students expecting to take the tests of the College Extrance Examination Board to procure the latest syllabus issued by the board.

APPLICATIONS

1. The principle that three points determine a plane is utilized in the tripod, which supports a camera or a surveyor's transit, and in the artist's easel. This form of support is adapted especially well to uneven surfaces. Is it necessary that the three legs be of the same length? Why?

2. A discussion arose in a certain school as to whether the flagpole on the school court was perpendicular to the court. Several pupils suggested that tests be made with a try-square. Explain how this should be done.

One of the pupils made use of the rope used for raising the flag. He stretched it taut and found that it reached the ground at a point 10 ft.



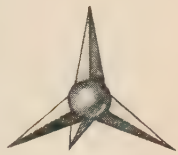
Courtesy Mackay Radio and Telegraph Co., Inc.

A RADIO COMPASS

(See p. 147)

6 in. from the base of the pole. In repeating the experiment he found that the distance varied from 10 ft. 6 in. to 10 ft. What conclusion do you draw from his measurements?

3. A caltrop is a small iron instrument with four prongs so arranged that if any three rest on a plane surface, the fourth is perpendicular to that surface. Caltrops were strewn over the ground in former warfare to impede the progress of cavalry or infantry. The jacks used by children in playing the game of jacks are of a similar formation except that they have six prongs with rounded ends.



What principle of geometry is made use of in the construction of a caltrop? What is the size of the angle between any two of the prongs of a caltrop? What is the size of the angle between any two of the prongs of a jack?

4. In a certain church where the organ was run by a water motor, the governing board decided that the one-inch pipe which supplied the motor was too small. They thought it would be advisable to double the power of the motor. "That means," said one, "that we should install a two-inch pipe." "No," objected a member of the board who understood geometry, "a two-inch pipe would be too big." Was he right and what should be approximately the size of a pipe with a cross section double in area the cross section of a one-inch pipe, assuming that the flow of water is in proportion to the areas of the cross sections.

5. Cylindrical forms are used in buildings to support heavy weights because they have a maximum breaking strength for a given volume. A hollow cylindrical iron post with an outside diameter of 10 in. and an inside diameter of 8 in. has the same volume as a solid iron post with a diameter of only 6 in. for any given length. Show how this result is found. Which post will support the greater weight?

In nature this principle is well illustrated in the trunks of bamboo trees and banana trees and in the stalks of sugar cane and corn. Give other examples of a similar nature from the vegetable kingdom.

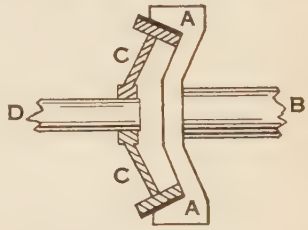
6. A company which puts out enormous quantities of its products in tin cans desired to find out the relation between the height of a can and the diameter of its base in order to have a maximum capacity with a given area. Accordingly its officers were asked to make calculations with varying values for h and d. If you were an official in that company, what report would you make? Why does a company not always use the form of can which has a maximum volume with a given area?

NOTE: This is a problem which can be solved without experiment by the methods of Differential Calculus.

7. For a certain sum a plumber took a contract to build a rectangular tank with a copper lining. The tank was to have a square base and an open top and a volume of 108 cubic feet. What should be its dimensions to require the least possible quantity of copper lining with a given thickness?

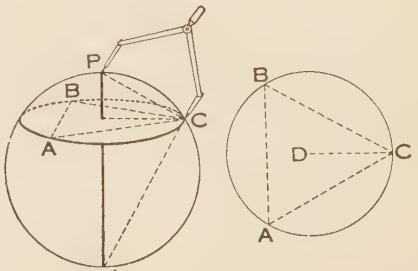
8. Cylinders in the form of water pipes are used to obtain a larger surface for a given quantity of material. This principle is employed in some types of boilers for the rapid heating of water. The steam and hot water radiators in our homes consist of a series of cylinders for quickly heating the air in the rooms; the radiators of our automobiles also consist of a series of cylinders for quickly cooling the water heated by the motor.

9. The form of a cone or a frustum of a cone was formerly used in the automobile in the cone clutch. A cross section of a cone clutch is shown in the illustration.

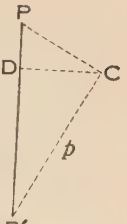


A is a cross section of the flywheel which is connected with the engine by shaft *B*. *C* is a cross section of the cone clutch which is connected by shaft *D* with the gears which drive the car. When the clutch pedal is in its regular position, the surface of the cone clutch is held tightly against the inner surface of the flywheel and the clutch turns with the flywheel. When the clutch pedal is depressed, the clutch is pulled out of contact with the flywheel. The modern car uses a multiple-disc clutch.

10. The diameter of a material sphere may be found in the following manner. Place one foot of the spherical compasses at any point (*P*) on the sphere and with a convenient opening such as *PC* draw a circle *ABC*. Measure with the compasses the three chords *AB*, *BC*, and *AC* and with them as sides construct

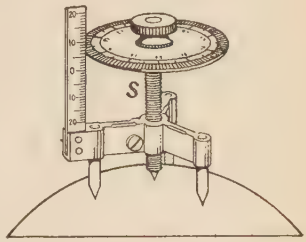


$\triangle ABC$ in a plane. Find DC, the radius of the circumscribed circle. (Describe the method.) Construct a right triangle with one leg equal to DC and the hypotenuse equal to PC. (Describe the method.) Draw $p \perp PC$ and produce PD to meet p at P'. PP' is the diameter of the sphere.



In the previous exercise, if PC is 10 in. and DC is 8 in., find the diameter of the sphere by calculation.

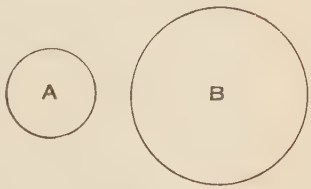
11. The radius of a sphere and the radius of curvature of a lens can be determined by an instrument called a spherometer. The three feet of the tripod determine a circle of known radius and the height of the foot of the screw S above the plane of the circle can be determined by readings from the vertical scale and a vernier on the disc.



12. Sound waves spread from their source in the form of an expanding sphere so long as their progress is not impeded by opposing objects. How does the area of such a sphere, all of whose points are 10 ft. from the source, compare in ratio with the area of a sphere whose points are 100 ft. from the source? How does the intensity of the sound at a point 10 ft. from the source compare with its intensity 100 ft. from the source? Why is the fact illustrated by this example called the law of inverse squares? Do you think the same law applies to light, heat, magnetic forces, and gravitation?

The pupil should note that in geometry the areas of similar solids, including spheres, vary directly as the squares of corresponding dimensions whereas in physics the intensities of certain kinds of waves and of other forces vary inversely as the squares of the distances from the source.

13. A and B in the figure represent two circular cross sections of solid cylinders of equal lengths. The diameter of B is double the diameter of A. If one foot of the smaller cylinder costs 30 cents, is a foot of the larger cylinder worth one dollar?



Let A and B represent also sections through the centers of two solid

spheres with the diameter of B double the diameter of A. If the smaller costs fifty cents, would you pay three dollars for the larger sphere? What are the values of the two larger solids in each case supposing the materials to be the same.

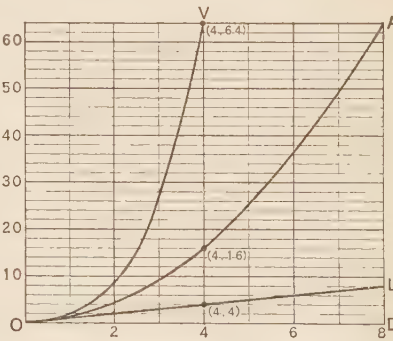
The eye often deceives us in our judgments of areas and volumes of similar solids. We do not always realize that an object which appears to be only a little larger than another may in reality be several times as large.

14. The use of spheres in ball bearings and of cylinders as rollers is illustrated in the Applications of Plane Geometry.

15. A discussion arose in a geometry class as to whether a twelve-inch globe or two circular flat maps each sixteen inches in diameter represented the surface of the earth with the greater detail. What is your decision?

16. What is the value of the gold in a gold basket-ball charm, in the form of a hollow sphere which is $\frac{3}{4}''$ in diameter and $\frac{1}{16}''$ thick, if one cubic inch of gold weighs 10.168 troy ounces and one ounce of gold is fixed by law at \$20.67183?

17. The accompanying graph shows the comparative increases and rates of increase for lines, areas, and volumes of a solid which expands so that at all stages the solids which are formed are similar to the original solid. The increase of any given dimension (OD) is represented on the horizontal scale from 0 to 8. The corresponding increase of any other line in the figure is represented by the line OL. The corresponding increases in areas and volumes are represented by the lines OA and OV respectively.



Observe the following important facts:

(*a*) The increase of any line or dimension OL varies directly as the increase of any other line or dimension OD and the rates of increase are constant;

(*b*) The increase in area varies as the square of any dimension and the rate of surface expansion becomes increasingly greater for the larger dimensions;

(*c*) The increase in volume varies as the cube of any dimension and

the rate of increase in volume is considerably greater than the rate of increase in dimensions as the dimensions become larger.

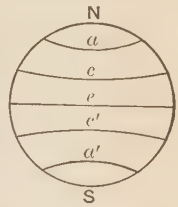
The following questions refer to a solid expanding as previously described so that all its forms are similar. The answers are to be determined approximately from the table.

1. If the length is increased $2\frac{1}{2}$ times, the area is increased how many times? The volume is increased how many times?

2. If the area is increased 16 times, the volume is increased how many times? Any dimension is increased how many times?

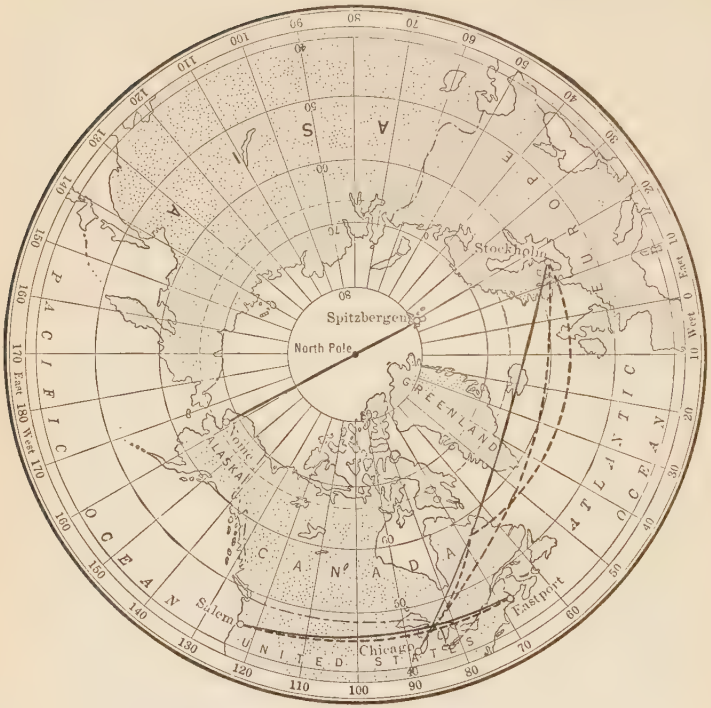
3. If the volume is increased 40 times, the area is increased how many times? Any dimension is increased how many times?

18. For convenience of geographical classification the surface of the earth is divided into six parts by the five circles represented in the figure. The terms employed in this classification have been studied by the pupil in geography.



N and S represent the north and south poles respectively and e is midway between them. What are the names given to circles e, c, c', a and a'? What is the surface called which is bounded by two consecutive circles? What is the solid called which is bounded by two consecutive circles? c is $23\frac{1}{2}$ degrees from e and a is $23\frac{1}{2}$ degrees from N. How many degrees are between c and c'? Are these circles established naturally or artificially? What geographical name is given to each of these six divisions of the earth's surface.

19. The center of gravity or centroid of a triangle was defined in Plane Geometry as the point of concurrence of the medians. The center of gravity of a circle is its center. It is stated in Plane Geometry that Pappus, the last important Greek mathematician, who lived about the end of the third century, discovered that if a closed plane figure is revolved about an external axis in its plane, (1) the area of the solid generated equals the product of the perimeter of the figure and the length of the circle generated by the centroid ($S = p \times 2\pi r$); (2) the volume of the solid generated equals the product of the area of the figure and the length of the circle generated by the centroid ($V = S \times 2\pi r$). If a circle revolves about an external axis in its plane the ring which is generated resembles in form the inner tube of an automobile tire. Find from the theorem of Pappus the area and the volume of an inner tube, when inflated, if the diameter of a cross section is 6 in. and the diameter of the circle generated by the center of a cross section is 26 in.



GREAT CIRCLE COURSES

In this map, the great circle courses are indicated by heavy solid lines; other courses are shown as dash lines. Note the favorable location of Nome and Spitzbergen for a trip through the Pole. The names of Byrd, Bennett, Amundsen, Ellsworth, and Nobile are associated with this course.

Observe that the great circle course from Chicago to Stockholm passes farther north than either city, that it is shorter than any other route, and that Greenland and Iceland offer "stepping stones across the Atlantic." The difference in the lengths of the paths from Eastport to Salem is discussed on p. 147.

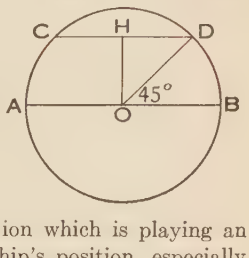
20. When we wish to find the shortest distance between two cities, we sometimes measure with a ruler the distance between them on a map and multiply the number of miles to the inch, according to the scale, by the number of inches. But on the spherical surface of the earth, a straight line is not the shortest line between the points. Let us take two cities, Eastport in Maine and Salem in Oregon, both of which are approximately on the parallel of latitude 45° North. We might think that the parallel of latitude 45° North running due west would mark the shortest distance from Eastport to Salem. But in solid geometry we can prove that the shortest line between two points on a sphere is the minor arc of a great circle through the points. A plane determined by the center of the earth and the cities of Eastport and Salem intersects the earth's surface in a great circle arc which runs at its mid-point over 100 miles north of the 45th parallel of latitude.

The length in degrees of such a great circle arc may be calculated by the Table of Natural Functions. (See page 154.) Let the number of degrees on the great circle arc between points X and Y be denoted by N, the latitude of X and Y by A and B respectively, and the difference in their longitude by C. By a formula of spherical trigonometry, $\cos N = \cos A \cos B \cos C + \sin A \sin B$. From this formula the great circle arc between Eastport and Salem is 38.77°. The length of 1° of a great circle arc of the earth is about 69.1 statute miles. (How is this result obtained?) Accordingly the length of the great circle arc between Eastport and Salem is approximately 2696.9 miles.

21. It may be interesting to know, for comparison with the result of the previous example, the length of the small circle arc between Eastport and Salem measured on the parallel of latitude 45° North.

Let the center of the earth be denoted by O, the diameter of the equator by AB, the diameter of a small circle 45° North by CD, and the radius of the earth (about 3956 miles) by OD. $\angle BOD = \angle D = 45°$.

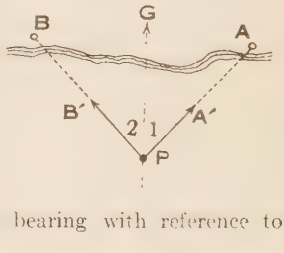
$\text{Cos } D = \dfrac{DH}{DO}$. $DH = 3956 \times .7071 = 2797.3$ miles. The circumference of the small circle arc CD = 17,583 miles. 1° of the small circle arc = 17,583 ÷ 360 = 48.84 miles. The small circle arc of 56° between Eastport and Salem = 2975 miles, which is about 278 miles greater than the great circle distance.



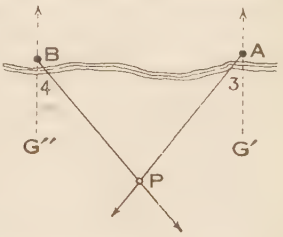
22. The radio compass is a modern invention which is playing an important part in navigation in locating a ship's position, especially when it is near land.

The radio compass is essentially a powerful receiving set with a large loop aërial. Anyone familiar with a loop aërial knows that by turning the loop, radio signals of varying intensity are obtained. When the loop is pointed directly towards a station, the signals from that station are the loudest and when the loop is at right angles to the line from that station, the signals are inaudible. It is easier to determine the position of zero intensity than the position of greatest intensity. Therefore the needle of the radio compass in the ship's pilot house is set at right angles to the large loop aërial in an inclosure above the pilot house. When the needle and loop, which are connected by a shaft which revolves with them, are rotated simultaneously until signals from a certain station are not heard, the loop is at right angles to the line from that station and the needle of the compass is pointing directly towards the station.

In plotting a ship's position from signals received by the radio compass, a navigator uses straight lines on his chart and not arcs of great circles because of the relatively small distances involved.



Suppose, for example, that a vessel is fog bound at point P and that A and B are radio transmitting stations on the coast. The line of direction PA' may be found as previously described. This line and the north-south line PG of the magnetic compass determine the angle or bearing ($\angle 1$) of the ship with reference to A. In like manner, $\angle 2$ determines the bearing with reference to station B.

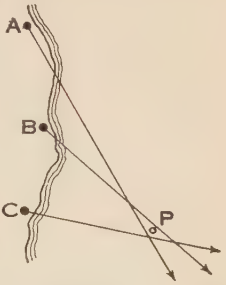


The navigator may now take a chart and with AG' as a north-south line through A draw $\angle 3 = \angle 1$. In like manner, he draws $\angle 4 = \angle 2$. The intersection of AP and BP determines the position of the ship on the chart.

In the method just described, the radio compass is on the ship and the stations at A and B serve as radio lighthouses. In other cases, the radio operator on a vessel calls station A, which is equipped with a radio compass, and station A gives the vessel its bearings. The operator, in like manner, obtains the bearings from stations B and C. The lines which are then drawn on a chart may not all pass through the

same point, but form a triangle called by navigators a "cocked hat." The center of the cocked hat is taken as the position of the vessel.

There are other interesting and valuable uses of this modern invention which might be described. To whatever use the radio compass may be put in the future, a knowledge of geometry will be essential.



RECREATIONS

1. Place six matches in such a position that they form four equilateral triangles in different planes. After you have done that, place six matches so that they form eight equilateral triangles in the same plane.

2. Tom said to Dick, "I have a problem in solid geometry for you. The area of a rectangular solid is 430 and its volume is 600. Find the dimensions." After Dick had worked on the problem a few minutes he exclaimed, "That problem is unfair." Is it, and if so, why?

3. In a room $9 \times 12 \times 15$ feet a wire is run from an upper corner to the diagonally opposite lower corner by being fastened to the walls or floor. Find the shortest length of the wire. Three pupils who worked the problem obtained for their answers 28.2, 26.8, and 25.8 feet.

4. Could you get a 7-in. cube through a cylinder whose inner diameter is 10 in.?

5. Can the total area of a cube be numerically equal to its volume? less than its volume? greater than its volume?

6. If you saw a wooden block in half, do you get a smaller block of half the original volume? of half the original area?

7. Is there any point on the earth's surface from which you could travel in only one direction? What is that direction?

8. A man traveled 15° north, then 15° east, then 15° south, and was back at his starting point. Can you explain?

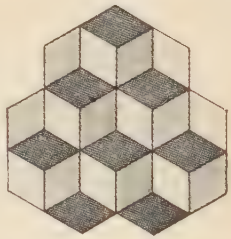
9. An old-time jingle puts a problem of solid geometry as follows:

"I agreed with a tinker whose name was Doolittle
To make for my aunt a flat-bottomed kettle.
Twelve inches exactly the depth of the same,
And sixteen gallons of ale to contain.
The inches across at the top would show
Just twice the width as measured below.
So tell me that width, across at the top,
For auntie now wants a lid from the shop."

10. Around a column twenty feet high and two feet in diameter a rope is wound exactly ten times from the top to the bottom. If the rope forms a uniform spiral, what is its length?

11. Which is the better bargain, 100 oranges three inches in diameter for \$2.90 or 80 oranges three and one-half inches in diameter for \$3.10?

12. Are there six or seven blocks in the annexed figure? Why does the number of blocks seem to change? How could the figure be altered so that only six blocks may be seen?



FORMULAS OF SOLID GEOMETRY

In learning the formulas for the areas of solids, it will be helpful for the pupil to note that all such expressions are of the *second degree*. The factor π is a constant and does not affect the degree of the equation.

In the formulas for lateral areas and volumes which follow, cylinders are grouped with prisms, cones with pyramids, and frustums of cones with frustums of pyramids. The two general formulas in each of these groups have a close correspondence, differing only in the fact that the bases of the rectilinear figures are polygons while the bases of the curvilinear figures are circles. The general formulas for the lateral areas of these figures may be recalled by association with the formulas for the areas of equivalent figures. It has been developed in previous exercises that the lateral surfaces of right prisms and cylinders are equivalent to rectangles, the lateral surfaces of regular pyramids and right circular cones are equivalent to triangles, and the lateral surfaces of frustums of a regular pyramid and a right circular cone are equivalent to trapezoids. From the general formula, in turn, the special formula (when its use seems desirable) may be derived by substitution and simplification.

The syllabus of the College Entrance Examination Board suggests the following: "Formulas for the areas of the surfaces of polyhedrons and for the total surface of a cone and cylinder should not be memorized. Every problem of this type should be solved by finding the sum of the areas of the constituent faces."

No formulas for total areas are given. Pupils should not try to memorize such formulas. In calculating a total area, the pupil should think of it as the sum of the areas of the base or bases and the lateral area.

The formulas for volumes are of the *third degree*. In such expressions as $\frac{1}{3}\ Bh$ and $\frac{1}{3}\ rS$, it should be noted that B, S, and M denote areas

and are second-degree forms. B, for example, may be the area of any polygon or the area of a circle.

The formulas for the volume of a sphere, a spherical pyramid, and a spherical sector are all of the pyramidal type, $V = \frac{1}{3} rS$. In this case, the altitude is the radius of the sphere (r) and the base S equals respectively the surface of the sphere, the area of a spherical polygon, and the area of a zone.

The prismatoid formula may be used for finding the volumes of all the figures in the following summary except the spherical pyramid and spherical sector. It is especially useful in finding the volumes of frustums and figures of that type.

Notation

A, B, C = angles of a lune and of spherical polygons.
B, B' = areas of lower and upper bases
c = circumference
E = spherical excess
h = altitude
l = slant height
M = area of midsection
p, p' = perimeters of lower and upper bases
r, r' = radii of lower and upper bases or radii of spheres
S = area of lateral surface, or area of a sphere, etc.
V = volume

Lateral Areas of Solid Figures (S)

Figure	General Formula	Special Formula
Right prism	ph	
Right circular cylinder	ch	$2\pi rh$
Regular pyramid	$\frac{1}{2} lp$	
Right circular cone	$\frac{1}{2} lc$	πrl
Frustum of regular pyramid	$\frac{1}{2} l(p + p')$	
Frustum of right circular cone	$\frac{1}{2} l(c + c')$	$\pi l(r + r')$
Sphere	$4\pi r^2$	
Zone	$2\pi rh$	
Lune	$\frac{A}{360} \cdot 4\pi r^2$	$\frac{A}{90} \cdot \pi r^2$
Spherical polygon	$\frac{E}{720} \cdot 4\pi r^2$,	$\frac{E}{180} \cdot \pi r^2$
Spherical excess	$A + B + C + \ldots \ldots - (n - 2)\ 180^\circ$	

Volumes of Solid Figures (V)

Figure	General Formula	Special Formula
Prism	Bh	
Circular cylinder	Bh	$\pi r^2 h$
Pyramid	$\frac{1}{3} Bh$	
Circular cone	$\frac{1}{3} Bh$	$\frac{1}{3} \pi r^2 h$
Frustum of pyramid	$\frac{1}{3} h(B + B' + \sqrt{B \cdot B'})$	
Frustum of circular cone	$\frac{1}{3} h(B + B' + \sqrt{B \cdot B'})$	$\frac{1}{3} \pi h(r^2 + r_1^2 + r r_1)$
Sphere	$\frac{1}{3} rS$	$\frac{4}{3} \pi r^3$
Spherical pyramid	$\frac{1}{3} rS$	
Spherical sector	$\frac{1}{3} rS$	$\frac{2}{3} \pi r^2 h$
Spherical wedge	$\frac{A}{360} \cdot \frac{4}{3} \pi r^3$	$\frac{A}{270} \cdot \pi r^3$
Spherical segment	$\frac{1}{6} \pi h(3 r^2 + 3 r_1^2 + h^2)$	
Prismatoid	$\frac{1}{6} h(B + B' + 4 M)$	

TABLE OF EQUIVALENTS

1 rod = $5\frac{1}{2}$ yds. = $16\frac{1}{2}$ ft. — 1 meter = 39.3700 inches

1 mile = 320 rd. = 5280 ft. — 1 inch = 2.5400 centimeters

1 acre = 43,560 sq. ft. = 160 sq. rd. = $\frac{1}{640}$ sq. mi.

1 pound (Av.) = .4536 kilogram — 1 kilogram = 2.2046 lb. (Av.)

1 gallon = 231.0000 cu. in. — 1 cu. ft. = 7.4805 gal.

1 bushel = 2150.4200 cu. in. — 1 cu. ft. = 0.8040 bu.

1 liter = 1.0567 liquid quarts = 0.9081 dry quart = 61.0250 cu. in.

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1 cu. ft. of water weighs approximately 62.5 pounds (Av.)

1 gallon of water weighs approximately 8.4 pounds (Av.)

1 liter of water weighs approximately 2.2 pounds (Av.)

$\pi = 3.1416$ (or 3.142 or 3.14 or 3.1 or $3\frac{1}{7}$); $1/\pi = 0.3183$; $\pi^2 = 9.8696$; $\sqrt{\pi} = 1.7725$; $1/\sqrt{\pi} = 0.5642$; $\pi^3 = 31.0063$; $\sqrt[3]{\pi} = 1.4646$; $\pi/4 = 0.7854$; $4\pi/3 = 4.1888$; $\sqrt{2} = 1.4142$; $\sqrt{3} = 1.7321$; $\sqrt{5} = 2.2361$.

Table of Powers and Roots

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

Table of Natural Functions

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875				
6°	.1045	.9945	.1051	50°	.7660	.6428	1.1918
7°	.1219	.9925	.1228	51°	.7771	.6293	1.2349
8°	.1392	.9903	.1405	52°	.7880	.6157	1.2799
9°	.1564	.9877	.1584	53°	.7986	.6018	1.3270
				54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679				
16°	.2756	.9613	.2867	60°	.8660	.5000	1.7321
17°	.2924	.9563	.3057	61°	.8746	.4848	1.8040
18°	.3090	.9511	.3249	62°	.8829	.4695	1.8807
19°	.3256	.9455	.3443	63°	.8910	.4540	1.9626
				64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663				
26°	.4384	.8988	.4877	70°	.9397	.3420	2.7475
27°	.4540	.8910	.5095	71°	.9455	.3256	2.9042
28°	.4695	.8829	.5317	72°	.9511	.3090	3.0777
29°	.4848	.8746	.5543	73°	.9563	.2924	3.2709
				74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002				
36°	.5878	.8090	.7265	80°	.9848	.1736	5.6713
37°	.6018	.7986	.7536	81°	.9877	.1564	6.3138
38°	.6157	.7880	.7813	82°	.9903	.1392	7.1154
39°	.6293	.7771	.8098	83°	.9925	.1219	8.1443
				84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	0.0000	∞

SUMMARY OF PLANE GEOMETRY

DEFINITIONS

This list of definitions is given for handy reference and for the purpose of aiding in rapid review work. The pupil should always give the definition of a term in a *complete sentence.*

Acute angle	An angle between 0° and 90°.
Adjacent angles	Two angles with a common vertex and a common side between them.
Alternate exterior angles	A pair of non-adjacent exterior angles on opposite sides of a transversal.
Alternate interior angles	A pair of non-adjacent interior angles on opposite sides of a transversal.
Altitude of a triangle	The perpendicular between a vertex and its opposite side.
Angle	The figure formed by two rays with common end points.
Apothem (regular polygon)	The radius of the circle inscribed in the polygon.
Arc	A portion of a circle.
Area of a surface	The number of square units of measure in the surface.
Broken line	A line composed of successive straight lines not in a straight line.
Center of a regular polygon	The center of its inscribed or circumscribed circle.
Central angle	An angle formed by two radii.
Chord	A line segment with its ends on a circle.
Circle	A closed curve everywhere equidistant from a point.
Circumference	The linear measure of a circle.
Circumscribed circle	One which passes through each vertex of a polygon.
Circumscribed polygon	One whose sides are tangent to a circle.
Complementary angles	Two angles whose sum equals 90°.
Concentric circles	Circles with the same center.
Concurrent lines	Lines which meet in a point.
Congruent figures	Figures which agree in shape and size.
Corresponding parts	Parts taken in the same order.
Cosine of an angle	In a right triangle, the ratio of the side adjacent to the angle to the hypotenuse.
Decagon	A polygon of ten sides.

Degree (angle)	$\frac{1}{90}$ of a right angle.
Degree (arc)	$\frac{1}{360}$ of a circle.
Diagonal	A line joining two non-consecutive vertices of a polygon.
Equiangular polygon	One whose angles are equal.
Equilateral polygon	One whose sides are equal.
Equivalent figures	Figures equal in area.
Exterior angle of a polygon	The angle included between a side of a polygon and an adjacent side produced.
Exterior interior angles	A pair of non-adjacent angles, the one exterior and the other interior on the same side of a transversal.
Hexagon	A polygon of six sides.
Hypotenuse	In a right triangle, the side opposite the right angle.
Inscribed angle	An angle formed by two chords intersecting on a circle.
Inscribed circle	A circle tangent to each side of a polygon.
Inscribed polygon	A polygon whose sides are chords of the same circle.
Isosceles trapezoid	One whose non-parallel sides are equal.
Isosceles triangle	One with two equal sides.
Locus of points	A place of points which answer certain conditions.
Mean proportion	A proportion with equal means.
Mean proportional	Either mean of a mean proportion.
Median (triangle)	The line joining a vertex to the middle of its opposite side.
Median (trapezoid)	The line joining the mid-points of the two non-parallel sides.
Oblique lines	Lines which form unequal adjacent angles.
Obtuse angle	An angle between 90° and 180°.
Octagon	A polygon of eight sides.
Parallel lines	Two lines in the same plane which have no common point if produced indefinitely.
Parallelogram	A quadrilateral whose opposite sides are parallel.
Pentagon	A polygon of five sides.
Perimeter	The linear measure of a polygon.
Perpendicular lines	Two lines which meet so that two equal adjacent angles are formed.
Polygon	A closed broken line.

Projection of a line upon another line	The segment of the second line between the feet of the perpendiculars dropped from the ends of the first line.
Proportion	The expression of equality of two fractions.
Quadrant	One-fourth of a circle.
Quadrilateral	A polygon of four sides.
Radius (circle)	A line from the center to the circle.
Radius (regular polygon)	The radius of its circumscribed circle.
Ratio	A fraction. The expressed division of one quantity by another.
Ray	A line which begins at a point and extends indefinitely in one direction from it.
Rectangle	A parallelogram with a right angle.
Regular polygon	An equilateral and equiangular polygon.
Rhombus	A parallelogram with two intersecting sides equal.
Right angle	An angle with perpendicular sides.
Right triangle	A triangle with one right angle.
Secant	A line passing through a circle in two points.
Sector	A figure bounded by two radii and their intercepted arc.
Segment of a circle	A figure bounded by a chord and its arc.
Semicircle	Half a circle.
Similar arcs	Arcs with the same valued central angles.
Similar polygons	Polygons which are mutually equiangular and whose corresponding sides are proportional.
Similar sectors	Sectors with the same valued central angles.
Similar segments	Segments with the same valued central angles.
Sine of an angle	In a right triangle, the ratio of the side opposite the angle to the hypotenuse.
Square	A rectangle with two intersecting sides equal.
Straight angle	An angle equal to double a right angle.
Supplementary angles	Two angles whose sum equals 180°.

Tangent of an angle	In a right triangle, the ratio of the side opposite the angle to the side adjacent to the angle.
Tangent to a circle	A line which touches a circle in only one place.
Tangent circles	Circles which are tangent to the same line at the same point.
Trapezoid	A quadrilateral with two sides parallel.
Triangle	A polygon of three sides.
Variable	A value which changes.
Vertical angles	A pair of non-adjacent angles formed when two lines intersect.

FACTS OF PLANE GEOMETRY

A. General Assumptions

1. Any magnitude equals itself. (Identity.)

2. Two magnitudes are equal if their ratio is unity, and conversely.

3. A magnitude may be substituted for its equal in an equation and in an inequality. (Substitution.)

4. Magnitudes equal to the same magnitude (or equal magnitudes) are equal.

5. A magnitude equals the sum of its parts.

6. If equals are added to equals, the *sums* are equal. (Axiom of Addition.)

7. If equals are subtracted from equals, the *differences* are equal. (Axiom of Subtraction.)

8. If equals are multiplied by equals, the *products* are equal. (Axiom of Multiplication.)

9. If equals are divided by equals, the *quotients* are equal. (The division by 0 is not permitted.) (Axiom of Division.)

NOTE: Special cases of Assumptions 8 and 9 are, "Doubles of equals are equal" and "Halves of equals are equal."

10. Equal powers or equal roots of equals produce *results* which are equal.

11. If unequals are operated upon by equals, the *results* are unequal in the same order.

12. If unequals are added to unequals of the same order, the *sums* are unequal in the same order.

13. Equal powers or equal roots of unequals produce *results* which are unequal in the same order.

14. If unequals are subtracted from equals, the *differences* are unequal in the reverse order.

15. A magnitude is greater than any of its parts.

16. If three quantities are so related that the first is greater than the second and the second is greater than the third, then the first is greater than the third.

17. Two quantities must possess one of three relations to each other; they are equal, or, the first is greater than the second, or, the first is less than the second.

18. Two lines can intersect in only one point.

19. A straight line intersects a circle in at most two points.

20. A straight line is the shortest line between two points.

21. The perpendicular from a point to a line is the shortest line between the point and the line.

22. If two adjacent angles are supplementary, their exterior sides form a straight line.

23. A point is within, on, or without a circle, according as its distance from the center is less than, equal to, or greater than a radius.

24. Through one point any number of straight lines can be drawn; and from a point any number of straight lines can be drawn equal to a given line.

25. A line segment can be produced from either end to any desired length.

26. A line segment can be divided into an integral number of equal parts. (A line can be bisected.)

27. One straight line and only one can be drawn from one point to another. (That is, two points determine a straight line.)

28. Through a point one line and only one can be drawn parallel to a given line. (Postulate of Parallels.)

29. Through a point one line and only one can be drawn perpendicular to a given line.

30. An angle can be constructed equal to a given angle.

31. An angle can be bisected.

32. One circle, and only one, can be drawn with any point as a center and with a radius of any given length.

33. Any geometric figure may be moved from one position to another without change either of shape or of size. (Postulate of Mobility.)

B. Congruent Triangles

Triangles are congruent, if they have

(1) *s.a.s.* = *s.a.s.* (3) *s.s.s.* = *s.s.s.*
(2) *a.s.a.* = *a.s.a.* (4) *s.a.a.* = *s.a.a.*

Right triangles are congruent, if they have

(5) *l.l.* = *l.l.* (7) *hy.l.* = *hy.l.*
(6) *l.a.* = *l.a.* (8) *hy.a.* = *hy.a.*

9. Either diagonal of a parallelogram divides it into congruent triangles.

C. Equal Lines

Lines are equal, if they are

(1) corresponding sides of congruent polygons;
(2) legs of an isosceles triangle;
(3) opposite sides of a parallelogram;
(4) equal to a third line or to equal lines;
(5) halves or doubles of the same line or equal lines;
(6) sums or differences of equal pairs of lines;
(7) sides opposite equal angles in the same triangle;
(8) sides of a regular polygon;
(9) perpendiculars to the sides of an angle from a point in the bisector of the angle;
(10) drawn to the extremities of a line from a point in the perpendicular bisector of the line;
(11) segments of intersecting diagonals of a parallelogram;
(12) segments of a side of a triangle formed by a line which bisects a second side and is parallel to a third side;
(13) legs of an isosceles trapezoid;
(14) segments of parallels cut off between parallels;
(15) drawn from any point in a perpendicular to a line cutting off equal distances from the foot of the perpendicular.

In the same circle (or in equal circles) **lines are equal,** if they are

(17) radii or diameters;
(18) chords with equal central angles;
(19) chords with equal arcs;
(20) chords equidistant from the center;
(21) perpendiculars from the center to equal chords;
(22) segments of a chord formed by a radius perpendicular to it;
(23) tangents from the same external point.

D. Parallel Lines

Lines are parallel, if

(1) they form equal alternate interior angles with the same transversal;
(2) they form equal alternate exterior angles with the same transversal;
(3) they form equal exterior-interior angles with the same transversal;
(4) they form two interior angles on the same side of a common transversal which are supplementary;
(5) they form two exterior angles on the same side of a common transversal which are supplementary;
(6) they are opposite sides of a parallelogram;
(7) they are perpendicular to or parallel to the same line;
(8) one is a side of a triangle and the other bisects the remaining sides;
(9) they are bases of a trapezoid;
(10) one is the common base of equal triangles and the other is a line through their vertices;
(11) one is a side of a triangle and the other divides the remaining sides proportionally.

E. Miscellaneous Line Relations

1. A line perpendicular to one of two parallel lines is perpendicular to the other.

2. A line which bisects two sides of a triangle is equal to one-half the third side.

3. The median of a trapezoid is parallel to the bases and equal to half their sum.

4. The hypotenuse of a 30-60-90 triangle is twice the side opposite the 30° angle.

5. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.

6. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides.

7. The altitudes of a triangle are concurrent.

8. The medians of a triangle are concurrent at a trisection point of each.

9. Any point equidistant from the extremities of a line lies in its perpendicular bisector.

10. Any point equidistant from the sides of an angle lies in the bisector of the angle.

11. Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.

12. A line through the center of a circle perpendicular to a chord bisects the chord and its two arcs.

13. A line through the center of a circle bisecting a chord (not a diameter) is perpendicular to the chord.

14. The perpendicular bisector of a chord passes through the center of the circle.

15. A line perpendicular to a radius at its outer extremity is tangent to the circle.

16. A tangent and a radius drawn to the point of tangency are perpendicular to each other.

17. A perpendicular to a tangent at the point of tangency passes through the center of the circle.

18. The line of centers of two tangent circles passes through the point of tangency.

19 The line of centers of two intersecting circles is the perpendicular bisector of their common chord.

20. Three points not in a straight line determine a circle.

21. If s is a side of a square and d its diagonal, $d = s\sqrt{2}$ and $s = \frac{1}{2}\, d\sqrt{2}$.

22. If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.

23. The product of the segments of one of two intersecting chords equals the product of the segments of the other.

24. The square of (or on) the hypotenuse of a right triangle equals the sum of the squares of (or on) the legs.

25. Two lines intersect, if the interior angles on the same side of a transversal to them are not supplementary.

26. Two lines intersect, if they are perpendicular to two intersecting lines.

F. Parallelograms

In a parallelogram

(1) the opposide sides are parallel;
(2) the opposite sides are equal;
(3) the opposite angles are equal;
(4) two consecutive angles are supplementary;

(5) the diagonals bisect each other;
(6) each diagonal forms two congruent triangles.

A quadrilateral is a parallelogram, if

(7) its opposite sides are parallel;
(8) its opposite sides are equal;
(9) one pair of sides is both equal and parallel;
(10) its diagonals bisect each other;
(11) its opposite angles are equal;
(12) any two pairs of consecutive angles are supplementary.

G. Equal Angles

Angles are equal, if they are

(1) corresponding angles of congruent polygons;
(2) angles opposite equal sides in the same triangle;
(3) opposite angles of a parallelogram;
(4) alternate interior angles of parallel lines;
(5) alternate exterior angles of parallel lines;
(6) exterior-interior angles of parallel lines;
(7) right angles; straight angles; vertical angles;
(8) equal to a third angle or to equal angles;
(9) halves or doubles of the same angle or of equal angles;
(10) complements or supplements of the same angle or of equal angles;
(11) the third angles of two triangles which have their other two angles respectively equal;
(12) angles whose sides are parallel (unless they are supplementary);
(13) angles whose sides are perpendicular (unless supplementary);
(14) angles of a regular polygon;
(15) corresponding angles of similar polygons.

In the same circle (or in equal circles) **angles are equal,** if they are

(16) central angles with equal arcs;
(17) central angles with equal chords;
(18) angles with their vertices on a circle and with the same or equal arcs;
(19) angles with equal numerical measures;
(20) angles inscribed in the same segment.

H. Miscellaneous Angle Relations

1. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

2. The sum of the angles about a point on the same side of a straight line equals a straight angle.

3. The line joining the vertices of two isosceles triangles on the same base bisects the angles at those vertices.

4. Interior angles on the same side of the transversal of parallel lines are supplementary.

5. The sum of the angles of a triangle equals a straight angle.

6. The angles adjacent to the hypotenuse of a right triangle are complementary.

7. An exterior angle of a triangle equals the sum of the non-adjacent interior angles.

8. In a right triangle whose hypotenuse is twice one of its legs, the angle opposite that leg is 30° and the triangle is a 30-60-90 triangle.

9. The sum of the interior angles of a polygon of n sides equals $(n-2)\ 180°$.

10. The sum of the exterior angles of a polygon of n sides, if the sides are produced in the same order, equals 360°.

11. Each interior angle of a regular polygon of n sides equals $\frac{(n-2)\ 180°}{n}$; and each exterior angle equals $\frac{360°}{n}$.

12. Tangents from a point to a circle make equal angles with the line from the point to the center.

13. The consecutive angles of a parallelogram are supplementary.

14. An angle inscribed in a semicircle is a right angle.

15. The central angle of a regular polygon of n sides equals $\frac{360°}{n}$.

I. Equal Arcs

In the same circle (or in equal circles) **arcs are equal,** if they are

(1) arcs of equal central angles;
(2) arcs of equal chords;
(3) arcs formed by a radius perpendicular to a chord;
(4) arcs cut off between two parallels;
(5) arcs with equal angles whose vertices are on the circle;
(6) arcs formed by a diameter.

7. Circles with equal radii are equal.

J. Angles Measured by Arcs

1. Central angles have the same ratio as their arcs.

2. A central angle equals numerically its intercepted arc.

3. An inscribed angle equals numerically one-half its intercepted arc.

4. An angle formed by two chords intersecting within a circle equals numerically one-half the sum of the arcs intercepted by it and its vertical angle.

5. An angle formed by a tangent and a chord equals numerically one-half the intercepted arc.

6. An angle formed by two secants, two tangents, or a tangent and a secant intersecting at an external point, equals numerically one-half the difference of the intercepted arcs.

K. Areas

The area of a

(1) **rectangle** = base $\times$ altitude;
(2) **square** = square of a side; half the square of a diagonal;
(3) **parallelogram** = base $\times$ altitude;
(4) **rhombus** = half the product of the diagonals;
(5) **triangle** = half the product of an altitude by its base;
(6) **trapezoid** = half the product of an altitude by the sum of the bases;
(7) **regular polygon** = half the product of its apothem by its perimeter.

L. General Properties of Proportions

1. A proportion may be transformed by Alternation, Inversion, Addition, Subtraction, and by any combination of them.

2. In any proportion the product of the extremes equals the product of the means.

3. If the product of two quantities equals the product of a second pair, either pair may be made the means and the other pair the extremes of a proportion.

4. A mean proportional between two quantities equals the square root of their product.

5. If three terms of two proportions are respectively equal, the fourth terms are equal.

6. In a series of equal ratios, the sum of the numerators and the sum of the denominators have the same ratio as any numerator and its denominator.

7. If the numerators of a proportion are equal, the denominators are equal, and conversely.

M. Ratios of Lines and Areas

1. Rectangles, parallelograms, and triangles in each case have the same ratio as the products of their bases by their altitudes.

2. Rectangles, parallelograms, and triangles in each case with equal bases and equal altitudes are equal.

3. Triangles with a common base and with their vertices in a line parallel to the base are equal.

4. Rectangles, parallelograms, or triangles with equal $\left\{\begin{matrix}\text{altitudes}\\\text{bases}\end{matrix}\right\}$ have the same ratio as their $\left\{\begin{matrix}\text{bases}\\\text{altitudes}\end{matrix}\right\}$.

5. Triangles with an angle of one equal to an angle of another have the same ratio as the products of the including sides.

6. A line parallel to one side of a triangle divides the other two sides in the same ratio; and conversely.

7. One side of a triangle and either segment cut off by a line parallel to a second side have the same ratio as the third side and its corresponding segment.

8. Corresponding segments cut off on two transversals by a series of parallel lines have the same ratio.

9. The bisector of an $\left\{\begin{matrix}\text{interior}\\\text{exterior}\end{matrix}\right\}$ angle of a triangle divides the opposide side $\left\{\begin{matrix}\text{internally}\\\text{externally}\end{matrix}\right\}$ in the same ratio as the other two sides.

10. Corresponding sides of similar polygons are proportional.

11. The perimeters of similar polygons have the same ratio as their corresponding sides.

12. Similar triangles have the same ratio as the squares of their corresponding sides.

13. Similar polygons have the same ratio as the squares of their corresponding sides.

14. A tangent from a point to a circle is the mean proportional between a secant from that point and its external segment.

15. The altitude upon the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.

16. Either leg of a right triangle is the mean proportional between the hypotenuse and its projection upon the hypotenuse.

17. The perpendicular from any point on a circle to a diameter is the mean proportional between the segments of the diameter; and the chord joining the point to either extremity of the diameter is the mean proportional between the diameter and its projection upon the diameter.

N. Similar Polygons

Triangles are similar, if they

(1) are mutually equiangular;

(2) have their respective sides parallel;

(3) have their respective sides perpendicular;
(4) have their respective sides proportional;
(5) have an angle of the one equal to an angle of the other and the including sides proportional;
(6) are similar to the same triangle;
(7) are corresponding triangles of similar polygons;
(8) have two angles of one equal to two angles of another;
(9) are right triangles with an acute angle of one equal to an acute angle of another.

Polygons are similar, if they

(10) are mutually equiangular and have their corresponding sides proportional;
(11) are composed of like-placed similar triangles;
(12) are regular polygons of the same number of sides.

13. The altitude upon the hypotenuse of a right triangle forms three pairs of similar triangles.

O. Regular Polygons and Circles

1. If a circle is divided into three or more equal arcs, the chords of these arcs form a regular inscribed polygon; the tangents at the points of division form a regular circumscribed polygon.

2. An equilateral polygon inscribed in a circle is regular.

3. Chords drawn from each vertex of a regular inscribed polygon to the mid-points of the adjacent arcs form a regular inscribed polygon of double the number of sides; tangents drawn at the mid-points of the arcs cut off by the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.

4. A chord whose length is that of a radius of a circle has an arc of $60°$ and is a side of a regular inscribed hexagon.

5. Regular hexagons can be inscribed in and circumscribed about a circle.

6. The radius of a regular polygon bisects an angle of the polygon and the apothem of a regular polygon is the perpendicular bisector of a side.

7. Regular polygons of the same number of sides are similar.

8. The perimeters of regular polygons of the same number of sides have the same ratio as their radii or as their apothems.

9. The areas of regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.

10. The area of a regular polygon equals half the product of the apothem by the perimeter.

11. Circles can be inscribed in and circumscribed about a regular polygon and such circles have a common center.

12. The circumferences of two circles have the same ratio as their radii or as their diameters.

13. The ratio of the circumference of a circle to its diameter equals a constant number called π.

14. The circumference of a circle equals π times the diameter.

15. The area of a circle equals π times the square of the radius.

16. The areas of two circles have the same ratio as the squares of their radii.

17. The area of a sector of a circle equals half the product of the radius by the length of its arc; or, $\frac{n}{360}$ times the area of the circle, where n is the number of degrees in its central angle.

18. The areas of similar sectors have the same ratio as the squares of their radii.

If the number of sides of an inscribed or circumscribed regular polygon is continuously increased *without bound*,

(19) the radius of the circle may be substituted in an equation for the apothem of the inscribed polygon;

(20) the circumference of the circle may be substituted in an equation for the perimeter of either polygon;

(21) the area of the circle may be substituted in an equation for the area of either polygon.

P. Inequalities

Lines are unequal

(1) if one is a straight line and the other is a broken line or a curved line between two points;

(2) if they are the hypotenuse and a leg of a right triangle;

(3) if they are sides of a triangle opposite unequal angles;

(4) if they are sides of two triangles with the other sides respectively equal, but the included angles unequal;

(5) if they are chords of a circle (or of equal circles) with unequal arcs;

(6) if they are chords of a circle (or of equal circles) unequally distant from the center;

(7) if they are perpendiculars from the center of a circle to unequal chords;

(8) if they are drawn to a line from any point in a perpendicular to that line and form unequal projections upon it;
(9) if they are projections upon a line formed by unequal lines drawn to that line from any point in a perpendicular to that line;
(10) if they are a diameter of a circle and any other chord.

Angles are unequal

(11) if they are an exterior angle and a non-adjacent interior angle of a triangle;
(12) if their complements or their supplements are unequal;
(13) if they lie opposite unequal sides of a triangle;
(14) if they are angles of two triangles with the including sides respectively equal, but the sides opposite them unequal;
(15) if they are central angles with unequal arcs.

Arcs of a circle (or of equal circles) **are unequal**

(16) if their central angles are unequal;
(17) if their chords are unequal.

Q. Loci

The locus of points

(1) equidistant from the extremities of a line is the perpendicular bisector of the line;
(2) equidistant from the sides of an angle is the bisector of the angle;
(3) equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by those lines;
(4) at a given distance d from a given point P is the circle having P as its center and d as its radius;
(5) at a given distance d from a given circle with radius r is two circles concentric with the given circle and with radii $r + d$ and $r - d$;
(6) equidistant from two concentric circles whose radii are R and r respectively is a concentric circle with a radius of $\frac{1}{2}(R + r)$;
(7) at a given distance d from a given line k is two lines parallel to and on opposite sides of k and d distance from it;
(8) equidistant from two parallel lines is the parallel mid-way between them.

FORMULAS OF PLANE GEOMETRY

Notation

a, b, c, . . . sides of polygon
b base of a figure
b, b' bases of a trapezoid
C circumference
D diameter
d diagonal of a square
h altitude
P perimeter
π 3.1416 or $3\frac{1}{7}$
R radius
r apothem
S area
s $\frac{1}{2}(a + b + c)$
s_n side of a regular polygon

Lengths of Lines

Right triangle, $c^2 = a^2 + b^2$.

Equilateral triangle, $h = \frac{s\sqrt{3}}{2}$.

$s_3 = R\sqrt{3}$.

Square, $d = s_4\sqrt{2}$. $s_4 = R\sqrt{2}$.

Hexagon, $r = \frac{R\sqrt{3}}{2}$, $s_6 = R$.

Circle $D = 2R$.

$C = 2\pi R = \pi D$.

Arc $= \frac{\text{central } \angle}{180} \times \pi R$.

Similar polygons,

$$\frac{P}{P'} = \frac{a}{a'} = \frac{b}{b'} = \text{etc.}$$

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \text{etc.}$$

$$\frac{a}{a'} = \frac{\sqrt{S}}{\sqrt{S'}}$$

Regular polygons of the same number of sides. $\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}$.

Circles, $\frac{R}{R'} = \frac{D}{D'} = \frac{\sqrt{S}}{\sqrt{S'}}$

Areas

Triangle, $S = \frac{1}{2}bh$.

$= \sqrt{s(s-a)(s-b)(s-c)}$

Equilateral triangle,

$S = \frac{1}{4}s^2 \times \sqrt{3}$.

Rectangle, $S = bh$.

Square, $S = s^2$.

Parallelogram, $S = bh$.

Trapezoid, $S = \frac{1}{2}h(b + b')$.

Regular polygon. $S = \frac{1}{2}rP$.

Circle, $S = \pi R^2 = \frac{1}{4}\pi D^2$.

Sector, $S = \frac{\text{central } \angle}{360} \times \pi R^2$.

$= \frac{1}{2}R \times arc$.

Segment,

$S = Sector \pm Triangle$.

Similar polygons,

$$\frac{S}{S'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \text{etc.}$$

Regular polygons of the same number of sides. $\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.

Circles, $\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{D^2}{D'^2}$.

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